MATHEMATICAL PSYCHICS

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AN ESSAY ON THE

APPLICATION OF MATHEMATICS TO

THE MORAL SCIENCES

BY

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INTRODUCTORY

DESCRIPTION OF

CONTENTS.

MATHEMATICAL PSYCHICS may be divided into two parts—Theoretical and Applied.

In the First Part (1) it is attempted to illustrate the possibility of Mathematical reasoning without *numerical* data (pp. 1–7); without more precise data than are afforded by estimates of *quantity of pleasure* (pp. 7–9). (2) An analogy is suggested between the *Principles of Greatest Happiness*, Utilitarian or Egoistic, which constitute the first principles of Ethics and Economics, and those *Principles of Maximum Energy* which are among the highest generalisations of Physics, and in virtue of which mathematical reasoning is applicable to physical phenomena quite as complex as human life (pp. 9–15).

The Calculus of Pleasure (Part II.) may be divided into two species—the Economical and the Utilitarian; the principle of division suggesting an addition to Mr. Sidgwick's 'ethical methods' (p. 16).

The first species of *Calculus* (if so ambitious a title may for brevity be applied to short studies in Mathematical Economics) is developed from certain *Definitions*

of leading conceptions, in particular of those connected with *Competition* (pp. 17–19). Then (α) a mathematical theory of *Contract unqualified by Competition* is given (pp. 20–30). (β) A mathematical theory of *Contract determined by Competition in a perfect Market* is given, or at least promised (pp. 30–33, and pp. 38–42). Reference is made to other mathematical theories of Market, and to Mr. Sidgwick's recent article on the 'Wages-Fund' (pp. 32, 33, and Appendix V.) (γ) attention is concentrated on the question—*What is a perfect Market?* It is argued that Market is imperfect, *Contract is indeterminate* in the following cases:—

(I.) When the number of competitors is limited (pp. 37, 39).

(II.) In a certain similar case likely to occur in contracts for *personal service* (pp. 42, 46).

(I. and II.) When the *articles* of contract are not perfectly divisible (p. 42, 46).

(III.) In case of *Combination*, Unionism; in which case it is submitted that (in general and abstractly speaking) *unionists stand to gain* in senses contradicted or ignored by distinguished economists (pp. 44, 47, 48).

(IV.) In a certain case similar to the last, and likely to occur in *Co-operative Association* (pp. 45, 49).

The *indeterminateness* likely from these causes to affect *Commercial Contracts*, and certainly affecting all sorts of *Political Contracts*, appears to postulate a *principle of arbitration* (pp. 50–52).

It is argued from mathematical considerations that *the basis of arbitration between contractors is the greatest possible utility of all concerned;* the Utilitarian first principle, which can of course afford only a general

direction—yet, as employed by Bentham's school, has afforded *some* direction in practical affairs (pp. 53–56).

The Economical thus leads up to the Utilitarian species of Hedonics; some studies in which already published[1] (under the title of 'Hedonical Calculus'—the species being designated by the generic title) are reprinted here by the kind permission of the Editor of 'Mind.'

Of the Utilitarian Calculus (pp. 56–82) the central conception is *Greatest Happiness*, the greatest possible sum-total of pleasure summed through all time and over all sentience. Mathematical reasonings are employed partly to confirm Mr. Sidgwick's proof that Greatest Happiness is the *end* of right action; partly to deduce middle axioms, *means* conducive to that end. This deduction is of a very abstract, perhaps only negative, character; negativing the assumption that *Equality* is necessarily implied in Utilitarianism. For, if sentients differ in *Capacity for happiness*—under similar circumstances some classes of sentients experiencing on an average more pleasure (*e.g.* of imagination and sympathy) and less pain (*e.g.* of fatigue) than others—there is no presumption that equality of circumstances is the most felicific arrangement; especially when account is taken of the interests of posterity.

Such are the principal topics handled in this *essay* or *tentative* study. Many of the topics, tersely treated in the main body of the work, are more fully illustrated in the course of seven supplementary chapters, or APPENDICES, entitled:

[1] *Mind*, July 1879.

Discussions too much broken up by this arrangement are re-united by references to the principal headings, in the Index; which also refers to the definitions of terms used in a technical sense. The Index also contains the names of many eminent men whose theories, bearing upon the subject, have been noticed in the course of these pages. Dissent has often been expressed. In so terse a composition it has not been possible always to express, what has always been felt, the deference due to the men and the diffidence proper to the subject.

MATHEMATICAL PSYCHICS.

ON THE APPLICATION OF MATHEMATICS TO THE MORAL SCIENCES.

THE application of mathematics to *Belief*, the calculus of Probabilities, has been treated by many distinguished writers; the calculus of *Feeling*, of Pleasure and Pain, is the less familiar, but not in reality[1] more paradoxical subject of this essay.

The subject divides itself into two parts; concerned respectively with principle and practice, root and fruit, the applicability and the application of Mathematics to Sociology.

PART I.

IN the first part it is attempted to prove an affinity between the moral and the admittedly mathematical sciences from their resemblance as to (1) a certain general complexion, (2) a particular salient feature.

(1) The science of quantity is not alien to the study of man, it will be generally admitted, in so far as actions and effective desires can be *numerically* measured by way of statistics—that is, very far, as Professor Jevons[2] anticipates. But in so far as our *data* may consist of

[1] Cf. Jevons, *Theory*, p. 9.
[2] Introduction to *Theory of Political Economy*.

estimates other than *numerical*, observations that some conditions are accompanied with *greater* or *less* pleasure than others, it is necessary to realise that mathematical reasoning is not, as commonly[1] supposed, limited to subjects where numerical data are attainable. Where there are data which, though not *numerical* are *quantitative*—for example, that a quantity is *greater* or *less* than another, *increases* or *decreases*, is *positive* or *negative*, a *maximum* or *minimum*, there mathematical reasoning is possible and may be indispensable. To take a trivial instance: a is greater than b, and b is greater than c, therefore a is greater than c. Here is mathematical reasoning applicable to quantities which may not be susceptible of numerical evaluation. The following instance is less trivial, analogous indeed to an important social problem. It is required to distribute a given quantity[2] of fuel, so as to obtain the greatest possible quantity of available energy, among a given set of engines, which differ in efficiency—*efficiency* being thus defined: one engine is more efficient than another if, whenever the total quantity of fuel consumed by the former is equal to that consumed by the latter, the total quantity of energy yielded by the former is greater than that yielded by the latter.

In the distribution, shall a larger portion of fuel be given to the more efficient engines? always, or only in some cases? and, if so, in what sort of cases? Here is a very simple problem involving no numerical data, yet

[1] The popular view pervades much of what Mill (in his *Logic*), after Comte, says about Mathematics applied to Sociology. There is a good expression of this view in the *Saturday Review* (on Professor Jevons's *Theory*, November 11, 1871.) The view adopted in these pages is expressed by Cournot, *Recherches*.)

[2] *Or*, a given quantity *per unit of time*, with corresponding modification of definition and problem.

requiring, it may be safely said, mathematics for its complete investigation.

The latter statement may be disputed in so far as such questions may be solved by reasoning, which, though not symbolical, is strictly mathematical; answered more informally, yet correctly, by undisciplined common sense. But, firstly, the advocate of mathematical reasoning in social science is not concerned to deny that mathematical reasoning in social, as well as in physical, science may be divested of symbol. Only it must be remembered that the question how far mathematics can with safety or propriety be divested of her peculiar costume is a very delicate question, only to be decided by the authority and in the presence of Mathematics herself. And, secondly, as to the sufficiency of common sense, the worst of such unsymbolic, at least unmethodic, calculations as we meet in popular economics is that they are apt to miss the characteristic advantages of deductive reasoning. He that will not verify his conclusions as far as possible by mathematics, as it were bringing the ingots of common sense to be assayed and coined at the mint of the sovereign science, will hardly realize the full value of what he holds, will want a measure of what it will be worth in however slightly altered circumstances, a means of conveying and making it current. When the given conditions are not sufficient to determinate the problem —a case of great importance in Political Economy— the ἀγεωμετρητὸς is less likely to suspect this deficiency, less competent to correct it by indicating what conditions are necessary and sufficient. All this is evident at a glance through the instrument of mathematics, but to the naked eye of common sense partially and ob-

scurely, and, as Plato says of unscientific knowledge, in a state between genuine Being and Not-Being.

The preceding problem, to distribute a given quantity of material in order to a maximum of energy, with its starting point *loose quantitative relations* rather than numerical data—its slippery though short path almost necessitating the support of mathematics—illustrates fairly well the problem of utilitarian distribution.[1] To illustrate the economical problem of exchange, the maze of many dealers contracting and competing with each other, it is possible to imagine[2] a mechanism of many parts where the law of motion, which particular part moves off with which, is not precisely given—with symbols, arbitrary functions, representing not merely *not numerical knowledge* but[3] *ignorance*—where, though the mode of motion towards equilibrium is indeterminate, the position of equilibrium is mathematically determined.

Examples not made to order, taken from the common stock of mathematical physics, will of course not fit so exactly. But they may be found in abundance, it is submitted, illustrating the property under consideration —mathematical reasoning without numerical data. In Hydrodynamics, for instance, we have a Thomson or Tait[4] reasoning 'principles' for 'determining P and Q *will be given later*. In the meantime it is obvious that *each decreases as X increases*. Hence the equations of motion show'—and he goes on to draw a conclusion of

[1] See p. 64. [2] See p. 34.

[3] *Ignoration of Co-ordinates* (Thomson and Tait, *Natural Philosophy*, 2nd edition), is appropriate in many social problems where we only know in part.

[4] Thomson and Tait, *Treatise on Natural Philosophy*, p. 320, 2nd edition. The italics, which are ours, call attention to the *unnumerical, loose quantitative, relation* which constitutes the datum of the mathematical reasoning.

momentous interest that balls (properly) projected in an infinite incompressible fluid will move as if they were attracted to each other. And generally in the higher Hydrodynamics, in that boundless ocean of perfect fluid, swum through by vortices, where the deep first principles of Physics are to be sought, is not a similar *unnumerical*, or *hyperarithmetical* method there pursued? If a portion of perfect fluid so moves at any time that each particle has no motion of rotation, then that portion of the fluid will retain that property for all time[1]; here is no application of the numerical measuring-rod.

No doubt it may be objected that these hydrodynamical problems employ some *precise* data; the very definition of Force, the conditions of fluidity and continuity. But so also have our social problems *some* precise data: for example, the property of *uniformity of price* in a market; or rather the (approximately realised) conditions of which that property is the deducible *effect*, and which bears a striking resemblance to the data of hydrodynamics:[2] (1) the *fulness* of the market: that there *continues* to be up to the conclusion of the dealing an indefinite number of dealers; (2) the *fluidity* of the market, or infinite dividedness of the dealers' interests. Given this property of uniform price, Mr. Marshall and M. Walras deduce mathematically, though not arithmetically, an interesting theorem, which Mill and Thornton failed with unaided reason to discern, though they were quite close to it—the theorem that the equation of supply to demand, though a necessary, is not a sufficient condition of market price.

To attempt to select representative instances from each

[1] Stokes, *Mathematical Papers*, p. 112.
[2] See p. 18.

recognised branch of mathematical inquiry would exceed the limits of this paper and the requirements of the argument. It must suffice, in conclusion, to direct attention to one species of Mathematics which seems largely affected with the property under consideration, the Calculus of Maxima and Minima, or (in a wide sense) *of Variations*. The criterion of a *maximum*[1] turns, not upon the *amount*, but upon the *sign* of a certain quantity.[2] We are continually concerned[3] with the ascertainment of a certain *loose quantitative relation*, the *decrease-of-rate-of-increase* of a quantity. Now, this is the very quantitative relation which it is proposed to employ in mathematical sociology; given in such data as the *law of diminishing returns to capital and labour*, the *law of diminishing utility*, the *law of increasing fatigue*; the very same irregular, unsquared material which constitutes the basis of the Economical and the Utilitarian Calculus.

Now, it is remarkable that the principal inquiries in Social Science may be viewed as *maximum-problems*. For Economics investigates the arrangements between agents each tending to his own *maximum* utility; and Politics and (Utilitarian) Ethics investigate the arrangements which conduce to the *maximum* sum total of

[1] *Maximum* in this paper is employed according to the context for (1) *Maximum* in the proper mathematical sense; (2) *Greatest possible*; (3) *stationary*; (4) where *minimum* (or *least possible*) might have been expected; upon the principle that every minimum is the correlative of a maximum. Thus Thomson's Minimum theorem is correlated with Bertrand's Maximum theorem. (Watson and Burbury.) This liberty is taken, not only for brevity, but also for the sake of a certain suggestiveness. '*Stationary*,' for instance, fails to suggest the *superlativeness* which it connotes.

[2] The second term of Variation. It may be objected that the *other* condition of a maximum equation of the first term to zero is of a more *precise* character. See, however, Appendix I., p. 92.

[3] E.g., Todhunter's *Researches on Calculus of Variations*, pp. 21–30, 80, 117, 286, &c.

utility. Since, then, Social Science, as compared with the Calculus of Variations, starts from similar data—*loose quantitative relations*—and travels to a similar conclusion—determination of *maximum*—why should it not pursue the same method, Mathematics?

There remains the objection that in Physical Calculus there is always (as in the example quoted above from Thomson and Tait) a potentiality, an expectation, of measurement; while Psychics want the first condition of calculation, *a unit*. The following[1] brief answer is diffidently offered.

Utility, as Professor Jevons[2] says, has two dimensions, *intensity* and *time*. The unit in each dimension is the just perceivable[3] increment. The implied equation to each other of each *minimum sensibile* is a first principle incapable of proof. It resembles the equation to each other of undistinguishable events or cases,[4] which constitutes the first principle of the mathematical calculus of *belief*. It is doubtless a principle acquired in the course of evolution. The implied equatability of time-intensity units, irrespective of distance in time and kind of pleasure, is still imperfectly evolved. Such is the unit of *economical* calculus.

For moral calculus a further dimension is required; to compare the happiness of one person with the happiness of another, and generally the happiness of groups of different members and different average happiness.

Such comparison can no longer be shirked, if there

[1] For a fuller discussion, see Appendix III.

[2] In reference to Economics, *Theory*, p. 51.

[3] Cf. Wundt, *Physiological Psychology*; below, p. 60. Our 'ebenmerklich' minim is to be regarded not as an infinitesimal differential, but as a finite small difference; a conception which is consistent with a (duly cautious) employment of infinitesimal notation.

[4] Laplace, *Essai—Probabilities*, p. 7.

is to be any systematic morality at all. It is postulated by distributive justice. It is postulated by the population question; that horizon in which every moral prospect terminates; which is presented to the far-seeing at every turn, on the most sacred and the most trivial occasions. You cannot spend sixpence utilitarianly, without having considered whether your action tends to increase the comfort of a limited number, or numbers with limited comfort; without having compared such alternative utilities.

In virtue of what *unit* is such comparison possible? It is here submitted: Any individual experiencing a unit of pleasure-intensity during a unit of time is to 'count for one.'[1] Utility, then, has *three* dimensions; a mass of utility, 'lot of pleasure,' is greater than another when it has more *intensity-time-number* units. The third dimension is doubtless an evolutional acquisition; and is still far from perfectly evolved.

Looking back at our triple scale, we find no peculiar difficulty about the third dimension. It is an affair of census. The second dimension is an affair of clock-work; assuming that the distinction here touched, between subjective and objective measure of time, is of minor importance. But the first dimension, where we leave the safe ground of the objective, equating to unity each *minimum sensibile*, presents indeed peculiar difficulties. *Atoms of pleasure* are not easy to distinguish and discern; more continuous than sand, more discrete than liquid; as it were nuclei of the just-perceivable, embedded in circumambient semi-consciousness.

We cannot *count* the golden sands of life; we cannot *number* the 'innumerable smile' of seas of love; but we

[1] In the Pure, for a *fraction*, in the Impure, imperfectly evolved, Utilitarianism. See p. 16.

seem to be capable of observing that there is here a *greater*, there a *less*, multitude of pleasure-units, mass of happiness; and that is enough.

(2) The application of mathematics to the world of soul is countenanced by the hypothesis (agreeable to the general hypothesis that every psychical phenomenon is the concomitant, and in some sense the other side of a physical phenomenon), the particular hypothesis adopted in these pages, that Pleasure is the concomitant of Energy. *Energy* may be regarded as the central idea of Mathematical Physics; *maximum energy* the object of the principal investigations in that science. By aid of this conception we reduce into scientific order physical phenomena, the complexity of which may be compared with the complexity which appears so formidable in Social Science.

Imagine a material Cosmos, a mechanism as composite as possible, and perplexed with all manner of wheels, pistons, parts, connections, and whose mazy complexity might far transcend in its entanglement the webs of thought and wiles of passion; nevertheless, if any given impulses be imparted to any definite points in the mechanism at rest, it is mathematically deducible that each part of the great whole will move off with a velocity such that the energy of the whole may be the greatest possible [1]—the greatest possible consistent with the given impulses and existing construction. If we know *something* about the construction of the mechanism, if it is 'a mighty maze, but not without a plan;' if we have some quantitative though not numerical datum about the construction, we may be able to deduce a similarly indefinite conclusion about the motion. For instance, any number of cases may be imagined in

[1] Bertrand's *Theorem*.

which, if a datum about the construction is that certain parts are *less stiff* than others, a conclusion about the motion would be that those parts[1] take on more energy than their stiffer fellows. This rough, indefinite, yet mathematical reasoning is analogous to the reasoning on a subsequent page,[2] that in order to the greatest possible sum total of happiness, the more capable of pleasure shall take more means, more happiness.

In the preceding illustration the motion of a mechanism was supposed instantaneously generated by the application of given impulses at definite points (or over definite surfaces); but similar general views are attainable in the not so dissimilar case in which we suppose motion generated in time by finite forces acting upon, and interacting between, the particles of which the mechanism is composed. This supposition includes the celebrated problem of Many Bodies (attracting each other according to any function of the distance); in reference to which one often hears it asked what can be expected from Mathematics in social science, when she is unable to solve the problem of Three Bodies in her own department. But Mathematics *can* solve the problem of many bodies—not indeed numerically and explicitly, but practically and philosophically, affording approximate measurements, and satisfying the soul of the philosopher with the grandest of generalisations. By a principle discovered or improved by Lagrange, each particle of the however complex whole is continually so moving that the accumulation of energy, which is constituted by adding to each other the energies of the mechanism existing at each instant of time (technically termed *Action*—the time-integral of Energy) should be a[3] maxi-

[1] Cf. Watson and Burbury, *Generalised Co-ordinates*, Art. 30, and preceding. [2] P. 64. [3] See note, p. 6.

mum. By the discovery of Sir William Rowan Hamilton[1] the subordination of the parts to the whole is more usefully expressed, the velocity of each part is regarded as derivable from the *action* of the whole; the action is connected by a *single*, although not an explicit or in general easily interpretable, relation with the given law of force. The many unknown are reduced to one unknown, the one unknown is connected with the known.

Now this accumulation (or time-integral) of energy which thus becomes the principal object of the physical investigation is analogous to that accumulation of pleasure which is constituted by bringing together in prospect the pleasure existing at each instant of time, the end of rational action, whether self-interested or benevolent. The central conception of Dynamics and (in virtue of pervading analogies it may be said) in general of Mathematical Physics is *other-sidedly identical* with the central conception of Ethics; and a solution practical and philosophical, although not numerical and precise, as it exists for the problem of the interaction of bodies, so is possible for the problem of the interaction of souls.

This general solution, it may be thought, at most is applicable to the utilitarian problem of which the object is the greatest possible sum total of universal happiness. But it deserves consideration that an object of Economics also, the arrangement to which contracting agents actuated only by self-interest tend is capable of being regarded upon the psychophysical hypothesis here entertained as the realisation of the maximum sum-total of happiness, the *relative maximum*,[2] or that which is *consistent with certain conditions*. There is dimly discerned the Divine idea of a power tending to

[1] *Philosophical Transactions*, 1834-5. [2] See pp. 24, 142.

the greatest possible quantity of happiness [1] *under conditions*; whether the condition of that perfect disintegration and unsympathetic isolation abstractedly assumed in Economics, or those intermediate [2] conditions of what Herbert Spencer might term integration on to that perfected utilitarian sympathy in which the pleasures of another are accounted equal with one's own. There are diversities of conditions, but one maximum-principle; many stages of evolution, but 'one increasing purpose.'

'Mécanique Sociale' may one day take her place along with 'Mécanique Celeste,' throned each upon the double-sided height of one maximum principle,[3] the supreme pinnacle of moral as of physical science. As the movements of each particle, constrained or loose, in a material cosmos are continually subordinated to one maximum sum-total of accumulated energy, so the movements of each soul, whether selfishly isolated or linked sympathetically, may continually be realising the maximum energy of pleasure, the Divine love of the universe.

'Mécanique Sociale,' in comparison with her elder sister, is less attractive to the vulgar worshipper in that she is discernible by the eye of faith alone. The statuesque beauty of the one is manifest; but the fairy-like features of the other and her fluent form are

[1] Cf. Mill, *Essays on Nature and Religion*. [2] See p. 16.

[3] The mathematical reader does not require to be reminded that upon the principles of Lagrange the whole of (conservative) Dynamics may be presented as a Maximum-Problem; if without gain, at any rate without loss. And the great principle of Thomson (Thomson & Tait, arts. Cf. *Theory of Vortices*, by Thomson, Royal Society, Edinburgh, 1865), with allied *maximum-principles*, dominating the theory of fluid motion, dominates Mathematical Physics with a more than nominal supremacy, and most indispensably efficacious power. Similarly, it may be conjectured, the ordinary moral rules are *equivalently* expressed by the Intuitivist in the (grammatically-speaking), *positive* degree, by the Utilitarian in the *superlative*. But for the higher moral problems the conception of *maximum* is indispensable.

veiled. But Mathematics has long walked by the evidence of things not seen in the world of atoms (the methods whereof, it may incidentally be remarked, statistical and rough, may illustrate the possibility of social mathematics). The invisible energy of electricity is grasped by the marvellous methods of Lagrange;[1] the invisible energy of pleasure may admit of a similar handling.

As in a system of conductors carrying electrical currents the energy due to electro-magnetic force is to be distinguished from the energy due to ordinary dynamical forces, *e.g.*, gravitation acting upon the conductors, so the energy of pleasure is to be distinguished not only from the gross energy of the limbs, but also from such nervous energy as either is not all represented in consciousness (*pace* G. H. Lewes), or is represented by *intensity of consciousness* not *intensity of pleasure*. As electro-magnetic force tends to a maximum energy, so also pleasure force tends to a maximum energy. The energy generated by pleasure force is the physical concomitant and measure of the conscious feeling of delight.

Imagine an electrical circuit consisting of two rails isolated from the earth connected at one extremity by a galvanic battery and bridged over at the other extremity by a steam-locomotive.[2] When a current of electricity is sent through the circuit, there is an electro-magnetic force tending to move the circuit or any moveable part of it in such a direction that the number of lines of force (due to the magnetism of the earth) passing through the circuit in a positive direction may be *a maximum*. The electro-magnetic force therefore tends to move the

[1] See Clerk Maxwell, *Electricity and Magnetism*, on the use of Lagrange's *Generalised Co-ordinates*, Part iv., chaps. 5 and 6.

[2] Clerk Maxwell has a similar construction.

locomotive along the rails in that direction. Now this delicate force may well be unable to move the ponderous locomotive, but it may be adequate to press a spring and turn a handle and let on steam and cause the locomotive to be moved by the steam-engine *in the direction of the electro-magnetic force*, either backwards or forwards according to the direction in which the electrical current flows. The delicate electro-magnetic force is placed in such a commanding position that she sways the movements of the steam-engine so as to satisfy her own yearning towards *maximum*.

Add now another degree of freedom; and let the steam-car governed move upon a *plane* [1] in a direction tending towards the position of Minimum Potential Electro-Magnetic Energy. Complicate this conception; modify it by substituting for the principle of Minimum Force-Potential the principle of *Minimum* [2] *Momentum-Potential*; imagine a comparatively gross mechanism of innumerable degrees of freedom *governed*, in the sense adumbrated, by a more delicate system—itself, however inconceivably diversified its degrees of freedom, obedient still to the great *Maximum Principles* of Physics, and amenable to mathematical demonstration, though at first sight as hopelessly incalculable as whatever is in life capricious and irregular—as the smiles of beauty and the waves of passion.

Similarly pleasure in the course of evolution has become throned among grosser subject energies—as it were explosive engines, ready [3] to go off at the pressure

[1] See p. 24.

[2] *Momentum-Potential* upon the analogy of *Velocity-Potential* (Thomson on Vortex Motion, § 31); and *Minimum*, as I venture to think, in virtue of certain analogies between theories about *Energy* and about *Action*.

[3] See the account of the *Mechanism of Life*, in Balfour Stewart's *Conservation of Energy*.

of a hair-spring. Swayed by the first principle, she sways the subject energies so as to satisfy her own yearning towards *maximum*; 'her every air Of gesture and least motion' a law of Force to governed systems—a fluent form, a Fairy Queen guiding a most complicated chariot, wheel within wheel, the 'speculative and active instruments,' the motor nerves, the limbs and the environment on which they act.

A system of such charioteers and chariots is what constitutes the object of Social Science. The attractions between the charioteer forces, the collisions and compacts between the chariots, present an appearance of quantitative regularity in the midst of bewildering complexity resembling in its general characters the field of electricity and magnetism. To construct a scientific hypothesis seems rather to surpass the powers of the writer than of Mathematics. 'Sin has ne possim naturæ accedere partes Frigidus obstiterit circum præcordia sanguis;' at least *the conception of Man as a pleasure machine* may justify and facilitate the employment of mechanical terms and Mathematical reasoning in social science.

PART II.

SUCH are some of the preliminary considerations by which emboldened we approach the two fields into which the Calculus of Pleasure may be subdivided, namely Economics and Utilitarian Ethics. The Economical Calculus investigates the equilibrium of a system of hedonic forces each tending to maximum individual utility; the Utilitarian Calculus, the equilibrium of a system in which each and all tend to maximum uni-

versal utility. The motives of the two species of agents correspond with Mr. Sidgwick's Egoistic and Universalistic Hedonism. But the correspondence is not perfect. For, firstly, upon the principle of 'self limitation' of a method, so clearly stated by Mr. Sidgwick, so persistently misunderstood by critics, the Pure Utilitarian might think it most beneficent to sink his benevolence towards competitors; and the *Deductive Egoist* might have need of a Utilitarian Calculus. But further, it is possible that the moral constitution of the concrete agent would be neither Pure Utilitarian nor Pure Egoistic, but μικτή τις. For it is submitted that Mr. Sidgwick's division of Hedonism—the class of 'Method' whose principle of action may be generically defined *maximising happiness*—is not exhaustive. For between the two extremes Pure Egoistic and Pure Universalistic there may be an indefinite number of impure methods; wherein the happiness of others as compared by the agent (in a calm moment) with his own, neither counts for nothing, not yet 'counts for one,' but *counts for a fraction.*

Deferring controversy,[1] let us glance at the elements of the *Economic Calculus*; observing that the *connotation* (and some of the reasoning) extends beyond the usual denotation; to the political struggle for power, as well as to the commercial struggle for wealth.

ECONOMICAL CALCULUS.

DEFINITIONS.—The first principle of Economics[2] is that every agent is actuated only by self-interest. The workings of this principle may be viewed under two aspects, according as the agent acts *without*, or *with*, the

[1] See Appendix IV.

[2] *Descriptions* rather, but sufficient for the purpose of these tentative studies.

consent of others affected by his actions. In wide senses, the first species of action may be called *war* ; the second, *contract.* Examples : (1) A general, or fencer, making moves, a dealer lowering price, *without consent of rival.* (2) A set of co-operatives (labourers, capitalists, manager) agreed *nem. con.* to distribute the joint-produce by assigning to each *a certain function* of his sacrifice. The *articles* of contract are in this case the *amount* of sacrifice to be made by each, *and the principle of distribution.*

'Is it peace or war?' asks the lover of 'Maud,' of economic *competition*, and answers hastily : It is both, *pax* or *pact* between contractors during contract, *war*, when some of the contractors *without the consent of others recontract.* Thus an auctioneer having been in contact with the last bidder (to sell at such a price *if* no higher bid) *recontracts* with a higher bidder. So a landlord on expiry of lease recontracts, it may be, with a new tenant.

The *field of competition* with reference to a contract, or contracts, under consideration consists of all the individuals who are willing and able to recontract about the articles under consideration. Thus in an auction the field consists of the auctioneer and all who are effectively willing to give a higher price than the last bid. In this case, as the transaction reaches determination, the field continually diminishes and ultimately vanishes. But this is not the case in general. Suppose a great number of auctions going on at the same point; or, what comes to the same thing, a market consisting of an indefinite number of dealers, say Xs, in commodity x, and an indefinite number of dealers, say Ys, in commodity y. In this case, up to the determination of equilibrium, the field continues indefinitely large. To

be sure it may be said to vanish at the position of equilibrium. But that circumstance does not stultify the definition. Thus, if one chose to define the *field of force* as the centres of force sensibly acting on a certain system of bodies, then in a continuous medium of attracting matter, the field might be continually of indefinite extent, might change as the system moved, might be said to vanish when the system reached equilibrium.

There is free communication throughout a *normal* competitive field. You might suppose the constituent individuals collected at a point, or connected by telephones—an ideal supposition, but sufficiently approximate to existence or tendency for the purposes of abstract science.

A *perfect* field of competition professes in addition certain properties peculiarly favourable to mathematical calculation; namely, a certain indefinite *multiplicity* and *dividedness*, analogous to that *infinity* and *infinitesimality* which facilitate so large a portion of Mathematical Physics (consider the theory of Atoms, and all applications of the Differential Calculus). The conditions of a *perfect* field are four; the first pair referrible[1] to the heading *multiplicity* or continuity, the second to *dividedness* or fluidity.

I. Any individual is free to *recontract* with any out of an indefinite number, *e.g.*, in the last example there are an indefinite number of Xs and similarly of Ys.

II. Any individual is free to *contract* (at the same time) with an indefinite number; *e.g.*, any X (and similarly Y) may deal with any number of Ys. This condition combined with the first appears to involve

[1] See p. 5.

the indefinite divisibility of[1] each *article* of contract (if any X deal with an indefinite number of Ys he must give each an indefinitely small portion of x); which might be erected into a separate condition.

III. Any individual is free to *recontract* with another independently of, *without the consent* being required of, any third party, *e.g.*, there is among the Ys (and similarly among the Xs) no *combination* or precontract between two or more contractors that none of them will recontract without the consent of all. Any Y then may accept the offer of any X irrespectively of other Ys.

IV. Any individual is free to *contract* with another independently of a third party; *e.g.*, in simple exchange each contract is between two only, but *secus* in the entangled contract described in the example (p. 17), where it may be a condition of production that there should be three at least to each bargain.

There will be observed a certain similarity between the relation of the first to the second condition, and that of the third to the fourth. The failure of the first involves the failure of the second, but not *vice versâ*; and the third and fourth are similarly related.

A *settlement* is a contract which cannot be varied with the consent of all the parties to it.

A *final settlement* is a settlement which cannot be varied by recontract within the field of competition.

Contract is *indeterminate* when there are an indefinite number of *final settlements*.

[1] This species of imperfection will not be explicitly treated here; partly because it is perhaps of secondary practical importance; and partly because it has been sufficiently treated by Prof. Jevons (*Theory*, pp. 135-137). It is important, as suggested in Appendix V., to distinguish the effects of this imperfection according as the competition is, or is not, supposed perfect in *other* respects.

The PROBLEM to which attention is specially directed in this introductory summary is: *How far contract is indeterminate*—an inquiry of more than theoretical importance, if it show not only that indeterminateness tends to prevent widely, but also in what direction an escape from its evils is to be sought.

DEMONSTRATIONS.[1]—The general answer is—(α) Contract without competition is indeterminate, (β) Contract with *perfect* competition is perfectly determinate, (γ) Contract with more or less perfect competition is less or more indeterminate.

(α) Let us commence with almost the simplest case of contract,—two individuals, X and Y, whose interest depends on two variable quantities, which they are agreed not to vary without mutual consent. Exchange of two commodities is a particular case of this kind of contract. Let x and y be the portions interchanged, as in Professor Jevons's example.[2] Then the utility of one party, say X, may be written $\Phi_1(a-x) + \Psi_1(y)$; and the utility of the other party, say Y, $\Phi_2(x) + \Psi_2(b-y)$; where Φ and Ψ are the integrals of Professor Jevons's symbols ϕ and ψ. It is agreed that x and y shall be varied only by consent (not *e.g.* by violence).

More generally. Let P, the utility of X, one party, $= \mathrm{F}(x\,y)$, and Π, the utility of Y, the other party, $= \Phi(x\,y)$. If now it is inquired at what point they will reach equilibrium, one or both refusing to move further, to what *settlement* they will consent; the answer is in general that contract by itself does not supply sufficient conditions to determinate the solution; supplementary conditions as will appear being supplied by

[1] *Conclusions* rather, the mathematical demonstration of which is not fully exhibited.

[2] *Theory of Political Economy*, 2nd ed., p. 107.

competition or ethical motives, Contract will supply only *one* condition (for the two variables), namely

$$\frac{dP}{dx}\frac{d\Pi}{dy} = \frac{dP}{dy}\frac{d\Pi}{dx}$$

(corresponding to Professor Jevons's equation

$$\frac{\phi_1(a-x)}{\psi_1(y)} = \frac{\phi_2(x)}{\psi_2(b-y)}$$

Theory p. 108), which it is proposed here to investigate.

Consider $P - F(xy) = 0$ as a surface, P denoting the length of the ordinate drawn from any point on the plane of $x\,y$ (say the plane of the paper) to the surface. Consider $\Pi - \Phi(xy)$ similarly. It is required to find a point (xy) such that, *in whatever direction* we take an infinitely small step, P and Π do not increase together, but that, while one increases, the other decreases. It may be shown from a variety of points of view that the locus of the required point is

$$\frac{dP}{dx}\frac{d\Pi}{dy} - \frac{dP}{dy}\frac{d\Pi}{dx} = 0\,;$$

which locus it is here proposed to call the *contract-curve.*

(1) Consider first in what directions X can take an indefinitely small step, say of length ρ, from any point $(x\,y)$. Since the addition to P is

$$\rho\left[\left(\frac{dP}{dx}\right)\cos\theta + \left(\frac{dP}{dy}\right)\sin\theta\right],$$

$\rho\cos\theta$ being $= dx$, and $\rho\sin\theta = dy$, it is evident that X will step only on one side of a certain line, the *line of indifference*, as it might be called; its equation being

$$(\zeta - x)\left(\frac{dP}{dx}\right) + (\eta - y)\left(\frac{dP}{dy}\right) = 0.$$

And it is to be observed, in passing, that the direction in which X will *prefer* to move, the line of force or *line of preference*, as it may be termed, is perpendicular to the line of indifference. Similar remarks apply to Π. If then we enquire in what directions X and Y will consent to move *together*, the answer is, in any direction between their respective lines of indifference, in a direction *positive* as it may be called *for both*. At what point then will they refuse to move at all? When their *lines of indifference* are coincident (and *lines of preference* not only coincident, but in opposite directions); whereof the *necessary* (but *not sufficient*) condition is

$$\left(\frac{d\mathrm{P}}{dx}\right)\left(\frac{d\Pi}{dy}\right)-\left(\frac{d\mathrm{P}}{dy}\right)\left(\frac{d\Pi}{dx}\right)=0.$$

(2) The same consideration might be thus put. Let the complete variation of P be $\mathrm{DP}=\rho\left[\left(\frac{d\mathrm{P}}{dx}\right)\cos\theta+\left(\frac{d\mathrm{P}}{dy}\right)\sin\theta\right]$ and similarly for Π. Then in general θ can be taken, so that $\frac{\mathrm{DP}}{\mathrm{D}\Pi}$ should be positive, say $=g^2$, and so P and Π both increase together.

$$\tan.\,\theta=-\frac{\frac{d\mathrm{P}}{dx}-g^2\frac{d\Pi}{dx}}{\frac{d\mathrm{P}}{dy}-g^2\frac{d\Pi}{dy}}$$

But this solution fails when $\dfrac{\left(\frac{d\mathrm{P}}{dx}\right)}{\left(\frac{d\mathrm{P}}{dy}\right)}=\dfrac{\left(\frac{d\Pi}{dx}\right)}{\left(\frac{d\Pi}{dy}\right)}$.

In fact, in this case $\frac{\mathrm{DP}}{\mathrm{D}\Pi}$ *is the same for all directions.*

If, then, that common value of $\frac{\mathrm{DP}}{\mathrm{D}\Pi}$ is *negative*, motion is impossible in any direction.

(3) Or, again, we may consider that motion is possible so long as, one party not losing, the other gains. The point of equilibrium, therefore, may be described as a *relative maximum*, the point at which *e.g.* Π being constant, P is a maximum. Put $\mathrm{P} = \mathrm{P} - c\,(\Pi - \Pi')$, where c is a constant and Π' is the supposed given value of Π. Then P is a maximum only when

$$dx\left(\frac{d\mathrm{P}}{dx} - c\,\frac{d\Pi}{dx}\right) + dy\left(\frac{d\mathrm{P}}{dy} - c\,\frac{d\Pi}{dy}\right) = 0\,;$$

whence we have as before the *contract-curve.*

The same result would follow if we supposed Y induced to consent to the variation, not merely by the guarantee that he should not lose, or gain infinitesimally, but by the understanding that he should gain sensibly with the gains of P. For instance, let $\Pi = k^2\mathrm{P}$ where k is a constant, certainly not a very practicable condition. Or, more generally, let P move subject to the condition that $\mathrm{DP} = \theta^2 \times \mathrm{D}\Pi$, where θ is a function of the coordinates. Then DP, *subject to this condition*, vanishes only when

$$0 = \left(\frac{d\mathrm{P}}{dx}\right)dx + \left(\frac{d\mathrm{P}}{dy}\right)dy + c\left\{\left(\frac{d\mathrm{P}}{dx}\right)dx + \left(\frac{d\mathrm{P}}{dy}\right)dy - \theta^2 \left[\left(\frac{d\Pi}{dx}\right)dx + \left(\frac{d\Pi}{dy}\right)dy\right]\right\}$$

where c is a constant;

$$\text{whence } \left(\frac{d\mathrm{P}}{dx}\right)\left(1 + c\right) - c\theta^2\left(\frac{d\Pi}{dx}\right) = 0$$

and $$\left(\frac{d\mathrm{P}}{dx}\right)\left(1 + c\right) - c\theta^2\left(\frac{d\Pi}{dy}\right) = 0\,;$$

whence as before $\left(\frac{dP}{dx}\right)\left(\frac{d\Pi}{dy}\right) - \left(\frac{dP}{dy}\right)\left(\frac{d\Pi}{dx}\right) = 0.$

No doubt the one theory which has been thus differently expressed could be presented by a professed mathematician more elegantly and scientifically. What appears to the writer the most philosophical presentation may be thus indicated.

(4) Upon the hypothesis above shadowed forth,[1] human action generally, and in particular the step taken by a contractor modifying articles of contract, may be regarded as the working of a gross force *governed*, let on, and directed by a more delicate pleasure-force. From which it seems to follow upon general dynamical principles applied to this special case that equilibrium is attained when the *total pleasure-energy of the contractors is a maximum relative*,[2] or subject, to conditions; the conditions being here (i) that the pleasure-energy of X and Y considered each as a function of (certain values of) the variables x and y should be functions of the *same* values: in the metaphorical language above employed that the charioteer-pleasures should drive their teams *together* over the plane of xy; (ii) that the joint-team should never be urged in a direction contrary to the *preference*[3] of either individual; that the resultant line of force (and the momentum) of the gross, the chariot, system should be continually intermediate between the (positive directions of the) lines of the respective pleasure-forces. [We may without disadvantage make abstraction of sensible momentum, and suppose the by the condition joint-system to move towards equilibrium along a line of resultant gross force. Let it start from the origin. And

[1] See pp. 13-15. [2] See note, p. 11. [3] See p. 22.

let us employ an *arbitrary function* to denote the unknown *principle of compromise* between the parties; suppose the ratio of the sines of angles made by the resultant line with the respective lines of pleasure-force.] Then, by reasoning different from the preceding only in the point of view, it appears that the *total utility of the system is a relative maximum at any point on the pure contract-curve.*

It appears from (1) and (2) there is a portion of the locus $\left(\frac{dP}{dx}\right)\left(\frac{d\Pi}{dy}\right) - \left(\frac{d\Pi}{dx}\right)\left(\frac{dP}{dy}\right) = 0$, where $\frac{DP}{D\Pi}$ is $+$, *not* therefore indicating immobility, *au contraire*, the im*pure* (part of the) contract-curve, as it might be called. This might be illustrated by two spheres, each having the plane of the paper as a diametral plane. The contract curve is easily seen to be the line joining the centres. Supposing that the distance between the centres is less than the less of the radii, part of the contract-curve is *impure.* If the index, as Mr. Marshall might call it, be placed anywhere in this portion it will run up to a centre. But between the centres the contract-curve is *pure*; the index placed anywhere in this portion is immovable; and if account be taken of the portions of the spheres underneath the plane of the paper, the downward ordinates representing *negative pleasures*, similar statements hold, *mutatis mutandis.*

It appears that the pure and impure parts of the contract-curve are demarcated by the points where $\frac{DP}{D\Pi}$ changes sign, that is (in general) where either $\frac{DP}{d\sigma}$ or $\frac{D\Pi}{d\sigma}$. ($d\sigma$ being an increment of the length of the contract-curve) either vanishes or becomes infinite. Accordingly the maxima and minima of P and Π present

demarcating points; for example, the centre of each sphere, which corresponds to a maximum in reference to the upper hemisphere, a minimum in reference to the lower hemisphere. The impure contract curve is relevant to cases where the commodity of one party is a *discommodity* to the other.

But even in the pure contract-curve all points do not in the same sense indicate immobility. For, according to the consideration (3) [above, p. 23], the contract-curve may be treated as the locus where, Π being constant, P is *stationary*, either a *maximum or minimum*. Thus any point in our case of two intersecting spheres affords a *maximum* in relation to the upper hemisphere; but the same point (it is only an accident that it should be *the same* point—it would not be the same point if you suppose slightly distorted spheres) affords a *minimum* in relation to the lower hemisphere. This *pure, but unstable* (part of the) contract-curve is exemplified in certain cases of that[1] *unstable equilibrium of trade*, which has been pointed out by Principal Marshall and Professor Walras.

The preceding theory may easily be extended to several persons and several variables. Let $P_1 = F_1 (x\,y\,z)$ denote the utility of one of three parties, utility depending on three variables, $x\,y\,z$; and similarly $P_2 = F_2$, $P_3 = F_3$. Then the *contract-settlement*, the arrangement for the alteration of which *the consent of all three parties* cannot be obtained, will be (subject to reservations analogous to those analysed in the preceding paragraphs) *the Eliminant.*

$$\frac{dP_1}{dx} \quad \frac{dP_1}{dy} \quad \frac{dP_1}{dz}$$

[1] Mr. Marshall's figure 9 but *not* his figure 8; for the delicate relation between the conceptions—instability of *Trade* (where *perfect competition* is presupposed) and instability of *contract in general*—is not one of identity.

$$\frac{dP_2}{dx} \quad \frac{dP_2}{dy} \quad \frac{dP_2}{dz}$$

$$\frac{dP_3}{dx} \quad \frac{dP_3}{dy} \quad \frac{dP_3}{dz}$$

In general let there be m contractors and n subjects of contract, n variables. Then by the principle (3) [above, p. 23] the state of equilibrium may be considered as such that the utility of any one contractor must be a maximum *relative to* the utilities of the other contractors being constant, or not decreasing; which may be thus mathematically expressed:

$D(l_1 P_1 + l_2 P_2 + \&c. + l_m P_m) = 0$, where D represents complete increment and l_1 l_2 &c., are indeterminate multipliers; whence, if there be n variables x_1 x_2 . . . x_n, we have n equations of the form

$$l_1 \frac{dP_1}{dx_1} + l_2 \frac{dP_2}{dx_1} + \&c. + l_m \frac{dP_m}{dx_1} = 0;$$

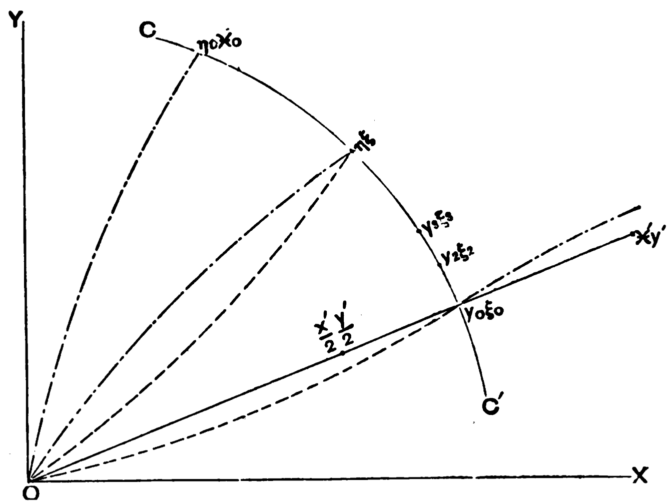
from which, if n be not less than m, we can eliminate the ($m-1$ independent) constants l and obtain the contract-system consisting of $n-(m-1)$ equations.

The case of n being less than m may be sufficiently illustrated by a particular example. Let the abscissa x represent the single variable on which the utilities P and Π of two persons contracting depend. Then if p and π are the maximum points for the respective pleasure-curves (compare the reasoning, p. 22) it is evident that the tract of abscissa between π and p is of the nature of pure contract-curve; that the index being placed anywhere in that tract will be immovable; secus on either side beyond π and p. Similarly it may be shown that, if three individuals are in contract about two variables x y, the contract locus or region is (the space within) a curvilinear triangle in the plane $x\,y$

bounded by the *three* contract-curves presented by successively supposing each pair of individuals to be in contract with respect to x and y. And similarly for larger numbers in hyperspace.

It is not necessary for the purpose of the present study to carry the analysis further. To gather up and fix our thoughts, let us imagine a simple case—Robinson Crusoe contracting with Friday. The *articles* of contract: wages to be given by the white, labour to be given by the black. Let Robinson Crusoe = X. Represent y, the labour given by Friday, by a horizontal line measured *northward* from an assumed point, and measure x, the remuneration given by Crusoe, from the same point along an *eastward* line (See accompanying figure 1.). Then

FIG. 1.



any point between these lines represents a contract. It will very generally be the interest of both parties to vary the articles of any contract taken at random. But there is a class of contracts to the variation of which the consent of *both* parties cannot be obtained, of *settle-*

ments. These settlements are represented by an *indefinite number* of points, a locus, the *contract-curve* CC′, or rather, a certain portion of it which may be supposed to be wholly in the space between our perpendicular lines in a direction trending from south-east to north-west. This available portion of the contract-curve lies between two points, say $\eta_0\, x_0$ north-west, and $y_0\, \xi_0$ south-east; which are respectively the intersections with the contract-curve of the *curves of indifference*[1] for each party drawn through the origin. Thus the utility of the contract represented by $\eta_0 x_0$ is for Friday zero, or rather, the same as if there was no contract. At that point he would as soon be off with the bargain—work by himself perhaps.

This simple case brings clearly into view the characteristic evil of indeterminate contract, *deadlock*, undecidable opposition of interests, ἀκριτὸς[2] ἔρις καὶ ταραχή. It is the interest of both parties that there should be *some settlement*, one of the contracts represented by the contract-curve between the limits. But *which* of these contracts is arbitrary in the absence of arbitration, the interests of the two *adversâ pugnantia fronte* all along the contract-curve, Y desiring to get as far as possible south-east towards $y_0\xi_0$, X north-west toward $\eta_0 x_0$. And it further appears from the preceding analysis that in the case of any number of *articles* (for instance, Robinson Crusoe to give Friday in the way of Industrial Partnership a *fraction* of the produce as well as wages, or again, *arrangements about the mode* of work), the *contract-locus* may still be represented as a sort of line, along which the pleasure-forces of the contractors are mutually antagonistic.

An accessory evil of indeterminate contract is the

[1] See p. 22. [2] Demosthenes, *De Coronâ.*

tendency, greater than in a full market, towards dissimulation and objectionable arts of higgling. As Professor Jevons[1] says with reference to a similar case, 'Such a transaction must be settled upon other than strictly economical grounds. . . . The art of bargaining consists in the buyer ascertaining the lowest price at which the seller is willing to part with his object, without disclosing, if possible, the highest price which he, the buyer, is willing to give.' Compare Courcelle-Seneuil's[2] account of the contract between a hunter and a woodman in an isolated region.

With this clogged and underground procedure is contrasted (β) the smooth machinery of the open market. As Courcelle-Seneuil says, 'à mesure que le nombre des concurrents augmente, les conditions d'échange deviennent plus nécessaires, plus impersonelles en quelque sorte.' You might suppose each dealer to write down[3] his *demand*, how much of an article he would take at each price, without attempting to conceal his requirements; and these data having been furnished to a sort of market-machine, the *price* to be passionlessly evaluated.

That contract in a state of perfect competition is determined by demand and supply is generally accepted, but is hardly to be fully understood without mathematics. The mathematics of a perfect market have been worked out by several eminent writers, in particular Messrs. Jevons, Marshall, Walras; to whose varied cultivation of the mathematical science, *Catallactics*, the reader is referred who wishes to dig down to the root of first principles, to trace out all the branches of a complete system, to gather fruits rare and only to be reached by a mathematical substructure.

[1] *Theory*, p. 134.
[2] *Traité*, book ii.
[3] Cf. Walras, *Elements*, Art. 50.

There emerges amidst the variety of construction and terminology *πολλῶν ὀνομάτων μορφὴ μία*, an essentially identical graphical form or analytical formula expressing the equation of supply to demand; whereof the simplest type, the catallactic molecule, as it might be called, is presented in the case above described in the definition of perfect competition.[1] The familiar pair of equations is deduced[2] by the present writer from the first principle: Equilibrium is attained when the existing contracts can neither be varied without recontract with the consent of the existing parties, nor by recontract within the field of competition. The advantage of this general method is that it is applicable to the particular cases of imperfect competition; where the conceptions of *demand and supply at a price* are no longer appropriate.

The catallactic molecule is compounded, when we suppose the Xs and Ys dealing in respect each of *several* articles with several sets of Zs, As, Bs, &c.; a case resolved by M. Walras.

Thus the actual commercial field might be represented by sets of entrepreneurs Xs, Ys, Zs, each X buying labour from among sets of labourers, As, Bs, Cs, use of capital from among sets of capitalists, Js, Ks, Ls, use of land from among sets of landowners, Ps, Qs, Rs, and selling products among a set of consumers consisting of the sum of the three aforesaid classes *and* the entrepreneurs of a species different from X, the Ys and Zs. As the demand of the labourer is deducible from considering his utility

[1] See p. 17. It must be carefully remembered that Prof. Jevons's Formulæ of Exchange apply not to bare individuals, an isolated couple, but (as he himself sufficiently indicates, p. 98), to individuals clothed with the properties of a market, a typical couple (see Appendix V.). The isolated couple, the catallactic *atom*, would obey our (a) law.

[2] See p. 38.

as a function of wages received and work done, so the demand of the entrepreneur is deducible from considering his utility as a function of (1) his expenditures on the agents of production; (2) his expenditures in the way of consumption; (3) his receipts from sale of produce; (4) his labour of superintendence. The last-named variable is not an article of contract; but there being supposed a definite relation connecting the produce with agents of production and entrepreneur's labour, the catallactic formulæ become applicable. This is a very abstract representation (abstracting *e.g.* risk, foreign trade, the migration from one employment to another, *e.q.* Xs becoming Ys,[1] &c.), yet more concrete than that of M. Walras, who apparently makes the more abstract supposition of a sort of *frictionless* entrepreneur, 'faisant[2] ni perte ni bénéfice.'

From the point of view just reached may with advantage be contemplated one of the domains most recently added to Economic Science—Mr. Sidgwick's contribution to the 'Fortnightly Review,' September, 1879. The *indirectness of the relation between wages and interest* which Mr. Sidgwick has so clearly demonstrated in words is self-evident in symbols. The *predeterminateness of the wage-fund*, which has received its *coup de grâce* from Mr. Sidgwick, must always, one would think, have appeared untenable from the humblest mathematical point of view, the consideration of the simplest type[3] of perfect competition; from which also it must be added that Mr. Sidgwick's — perhaps inadvertent, perhaps here misinterpreted—statement,[4] that contract

[1] This *permeability* between employments (such as explained in *Economics of Industry* with reference to the supply of unskilled and skilled labour and of business power) tends to a *level* of utility.

[2] *Elements*, Arts. 231, 242, &c.

[3] See pp. 17, 31.

[4] *Fortnightly Review*, 1879, pp. 410 (end) 411 (beginning).

between employer and operative even in the case of what is here called [1] *perfect* competition, is indeterminate, does not, it is submitted, appear tenable. It is further submitted that Mr. Sidgwick's strictures [2] on Prof. Jevons are hasty; for that by a (compound) employment of the Jevonian (or an equivalent catallactic) formula, the complex relations between entrepreneur, capitalist, and labourer are best made clear. And so 'there *is à priori* ground for supposing that industrial competition tends to equalize the rate of *profit* (as well as *interest*) on capitals of different amount.'[3] That 'the labour of managing capital does not increase in proportion to the amount managed' is so far from creating any peculiar difficulty, that it is rather of the essence of the theory of exchange; quite congruent with the familiar circumstance that the *disutility* of (common) labour (labour subjectively estimated) does not increase in proportion to *work done* (labour objectively estimated). That the labour of managing capital increases not only *not at the same* but at a *less* rate-of-increase than the amount managed, as Mr. Sidgwick seems to imply, is indeed a peculiar circumstance; but it is of a sort with which the Jevonian formula, the mathematical theory of catallactics, is quite competent to deal, with which in fact Mr. Marshall has dealt in his *second class* of Demand-Curves.

[1] See Defin, p. 18.

[2] *Fortnightly Review*, pp. 411, 412.

[3] As the gain per unit of produce is the same for one X as for another X, and the gain per unit of capital lent is the same for one J as for another J; so, if there is in the field in addition to the classes prescinded, a class of capitalist-entrepreneurs, *e.g.* (J K)s, the gain per unit of produce is the same for one (J K) as for another (J K). But no equation is made between the gain of a (J K) and the sum of the gains of a J and a K; even if to simplify the comparison we abstract rent. (*Gain* of course in this statement measured objectively, say in money, not subjectively in utility).

But it is not the purport of the present study to attempt a detailed, much less a polemical, discussion of pure Catallactics, but rather (γ) to inquire how far contract is determinate in cases of imperfect competition. It is not necessary for this purpose to attack the *general problem of Contract qualified by Competition*, which is much more difficult than the general problem of unqualified contract already treated. It is not necessary to resolve analytically the composite mechanism of a *competitive field*. It will suffice to proceed synthetically, observing in a simple typical case the effect of continually introducing into the field additional competitors.

I. Let us start, then, from the abstract typical case above put (p. 28), an X and Y dealing respectively in x and y. Here x represents the *sacrifice objectively measured* of X; it may be manual work done, or commodity manufactured, or capital abstained from during a certain time. And y is the objectively measured remuneration of X. Hence it may be assumed, according to the two first axioms [1] of the Utilitarian Calculus, the law of increasing labour, and the law of decreasing utility, that P being the utility of X, (1)[2] $\frac{dP}{dx}$ is continually *negative*, $\frac{dP}{dy}$ *positive*; (2) $\frac{d_2P}{dx^2}$, $\frac{d_2P}{dy^2}$, $\frac{d_2P}{dx\,dy}$, continually *negative*. (Attention is solicited to the interpretation of the third condition.) No doubt these latter conditions are subject to many exceptions, especially in regard to abstinence from capital, and in case of pur-

[1] See these laws stated in the companion calculus. The proofs were offered in *Mind*, without acknowledgment, because without knowledge, of the cumulative proofs already adduced by Prof. Jevons.

[2] Cf. Appendix V.

chase not for consumption, but with a view to re-sale; and in the sort of cases comprised in Mr. Marshall's Class II. curves. Still, these exceptions, though they destroy the watertightness of many of the reasonings in this and the companion calculus, are yet perhaps of secondary importance to one taking a general abstract view.

This being premised, let us now introduce a second X and a second Y; so that the field of competition consists of two Xs and two Ys. And for the sake of illustration (not of the argument) let us suppose that the new X has the same requirements, the same nature as the old X; and similarly that the new Y is equal-natured with the old.

Then it is evident that there cannot be equilibrium unless (1) all the field is collected at one point; (2) that point is on the *contract-curve*. For (1) if possible let one couple be at one point, and another couple at another point. It will generally be the interest of the X of one couple and the Y of the other to rush together, leaving their partners in the lurch. And (2) if the common point is not on the contract-curve, it will be the interest of *all parties* to descend to the contract-curve.

The points of the contract-curve in the immediate neighbourhood of the limits $y_0\xi_0$ and $\eta_0 x_0$ cannot be *final settlements*. For if the system be placed at such a point, say slightly north-west of $y_0\xi_0$, it will in general be possible for *one* of the Ys (without the consent of the other) to *recontract* with the two Xs, so that for all those three parties the recontract is more advantageous than the previously existing contract. For the right line joining the origin to (the neighbourhood of) $y_0\xi_0$ will in general lie altogether within the *indifference-curve* drawn from the origin to $y_0\xi_0$. For the indif-

ference-curve is in general convex to the abscissa. For its differential equation is

$$-\frac{dy}{dx} = \frac{\left(\frac{dF(xy)}{dx}\right)}{\left(\frac{dF(xy)}{dy}\right)}$$

whence $$\frac{d_2y}{dx^2} =$$

$$\frac{-\left[\left(\frac{d_2F}{dx^2}\right) + \left(\frac{d_2F}{dxdy}\right)\frac{dy}{dx}\right]\left(\frac{dF}{dy}\right) + \left(\frac{dF}{dx}\right)\left[\left(\frac{d_2F}{dxdy}\right) + \frac{d_2F}{dy^2}\frac{dy}{dx}\right]}{\left(\frac{dF}{dy}\right)^2}$$

which is perfectly *positive*. Therefore the indifference-curve (so far as we are concerned with it) is convex to the abscissa.

Now, at the contract-curve the two indifference-curves for X and Y *touch*. Thus the figure 1, page 28, is proved to be a correct representation, indicating that a point $x'y'$ can be found both more advantageous for Y than the point on the contract-curve $y_1\xi_1$ (on an *interior* indifference-curve, as it may be said), and also such that its co-ordinates are the sums (respectively) of the co-ordinates of two other points, both more advantageous for an X. These latter points to be occupied by X_1 and X_2 may be properly regarded (owing to the symmetry and competition) as *coincident*; with co-ordinates $\frac{x'}{2}\ \frac{y'}{2}$. Further, it appears from previous reasonings that there will be a *contract-relation* between $(x'y')$ and $\left(\frac{x'}{2}\ \frac{y'}{2}\right)$;

namely $\frac{\Phi'_x\,(x'\ y')}{\Phi'_y\,(x'\ y')} = \frac{F'_x\left(\frac{x'}{2}\ \frac{y'}{2}\right)}{F'_y\left(\frac{x'}{2}\ \frac{y'}{2}\right)}$; where F'_x is put for the

first partially derived function $\left(\frac{d\,\mathrm{F}\,(x\,y)}{d\,x}\right)$

When this relation is satisfied the system of three might remain in the position reached; but for Y_2 who has been left out in the cold. He will now strike in, with the result that the system will be worked down to the contract-curve again; to a point at least as favourable for the Xs as $\frac{x'}{2}\,\frac{y'}{2}$. Thus the Ys will have lost some of their original advantage by competition. And a certain process of which this is an abstract typical representation will go on as long as it is possible to find a point $x'\,y'$ with the requisite properties. Attention to the problem will show that the process will come to a stop at a point on the contract-curve $y_2\,\xi_2$, such that if a line joining it to the origin intersect the curve, the *supplementary contract-curve* as it might be called,

$$\frac{\Phi'_x\,(x\,y)}{\Phi'_y\,(x\,y)} = \frac{\mathrm{F}'_x\left(\frac{x}{2}\,\frac{y}{2}\right)}{\mathrm{F}'_y\,\left(\frac{x}{2}\,\frac{y}{2}\right)}$$

in the point $x'\,y'$ then $\Phi\,(\xi_2\,y_2) = \Phi\,(x'\,y')$, *provided that* $\left(\frac{x'}{2}\,\frac{y'}{2}\right)$ falls within the indifference-curve for Y drawn through $(\xi_2\,y_2)$. If otherwise, a slightly different system of equations must be employed.

If now a *third* X and third Y (still equal-natured) be introduced into the field, the system can be worked down to a point $\xi_3\,y_3$;[1] whose conditions are obtained from those just written by substituting for $\frac{x'}{2}\,\frac{y'}{2}, \frac{2\,x'}{3}\,\frac{2\,y'}{3}$. For this represents the last point at which 2 Ys can recontract with 3 Xs with advantage to all five. Analyti-

[1] Compare the analysis in Appendix VII.

cal geometry will show that this point is lower down (in respect of the advantage of Y) than $\xi_2\ y_2$. In the limit, when the Xs and Ys are indefinitely (equally) multiplied, we shall have $(x'\ y')$ coincident with $(\xi_\infty\ y_\infty)$, or as we may say for convenience $(\xi\ \eta)$, satisfying one or other of the *alternatives* corresponding to those just mentioned.

In case of the first alternative we have

$$\xi\,\Phi'_x\,(\xi\,\eta) + \eta\,\Phi'_y\,(\xi\,\eta) = 0$$

For $\Phi\,(\xi\,\eta) = \Phi\,(x'\ y') = \Phi\,(\,(1+h)\,\xi\,(1+h)\,\eta)$. In the limiting case h is infinitesimal. Whence by differentiating the above equation is obtained. And the second alternative $\left(\frac{x'}{2}\ \frac{y'}{2}\right.$ *not* falling within the indifference-curve of Y) is not to be distinguished from the first in the limiting case.

If this reasoning does not seem satisfactory, it would be possible to give a more formal proof; bringing out the important result that the common tangent to both indifference-curves at the point $\xi\ \eta$ is the vector from the origin.

By a parity of reasoning it may be shown that, if the system had been started at the north-west extremity of (the available portion of) the contract-curve, it would have been worked down by competition *between the* Xs to the same point; determined by the intersection with the contract-curve of $\xi\,F'\,x + \eta\,F'_y = 0$; for the *same* point is determined by the intersection of *either* curve with the contract-curve. For the three curves evidently intersect in the same point.

Taking account of the two processes which have been described, the competing Ys being worked down for a certain distance towards the north-west, and similarly the competing Xs towards the south-east: we see that

in general for any number short of the *practically infinite* (if such a term be allowed) there is a finite length of contract-curve, from $\xi_m\ y_m$ to $x_m\ \eta_m$, at any point of which if the system is placed, it cannot by contract or recontract be displaced; that there are *an indefinite number of final settlements*, a quantity continually diminishing as we approach a perfect market. We are brought back again to case (β), on which some further remarks have been conveniently postponed to this place. (For additional illustrations see Appendix V.)

The two conditions, $\xi\,\Phi'x + \eta\,\Phi'_y = 0$ and $\xi F'_x + \eta F'_y = 0$, just obtained correspond to Professor Jevons's two equations of exchange. His formulæ are to be regarded as representing the transactions of two *individuals in, or subject to, the law of, a market.* Our assumed *unity of nature* in the midst of plurality of persons naturally brings out the same result. The represented two curves may be called *demand-curves*, as each expresses the amount of dealing which will afford to one of the dealers the maximum of advantage *at a certain rate of exchange a value of* $\frac{y}{x}$. This might be elegantly expressed in polar co-ordinates, tan θ will then be the rate of exchange, and, if P be the utility of X, $\left(\frac{dP}{d\rho}\right) = 0$ is the demand-curve. By a well known property of analysis $\left(\frac{dP}{d\rho}\right) = 0$ represents not only maximum points, but *minimum points*; the lowest depths of valley, as well as the highest elevations, which one moving continually in a fixed right line from the origin over the *utility-surface* would reach. This minimum portion of the demand-curve corresponds to Mr. Marshall's Class II. We see that the dealer at any given

rate of exchange, far from resting and having his end at a point on this part of the curve, will tend to move away from it. It has not the properties of a genuine demand-curve.

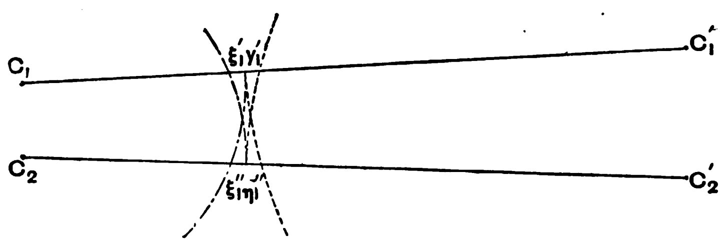
The dealing of an individual in an open market, in which there prevails what may be called the law of price, the relation between the individual's requirements and that quantity *collectively*-demanded-at-a-price, usually designated by the term *Demand*, between little *d* and big D in M. Walras's terminology, is elegantly exhibited by that author. Compare also Cournot on 'Concurrence.'

Here it is attempted to proceed without postulating the phenomenon of uniformity of price[1] by the longer route of *contract-curve*. When we suppose plurality of natures as well as persons, we have to suppose a plurality of contract-curves (which may be appropriately conceived as grouped, according to the well-known logarithmic law, about an average). Then, by considerations analogous to those already employed, it may appear that the quantity of final settlements is diminished as the number of competitors is increased. To facilitate conception, let us suppose that the field consists of two Xs, not equally, but nearly equally, natured; and of two Ys similarly related. And (as in the fifth Appendix) let the indifference curves consist of families of concentric circles. Then, instead of a single contract-curve, we have a contract-region, or bundle of contract-curves; namely the four lines joining the centres of the circle-systems, the lines $C_1C'_1$, $C_1C'_2$, $C_2C'_1$, $C_2C'_2$; wherein C_1, C_2 are the centres of X_1 and X_2, supposed close together; and similarly C'_1 and C'_2 for the Ys.

[1] The term will sometimes be used here for *rate of exchange in general*, as by M. Walras.

What corresponds here to that *settlement of the whole field at a single point in the contract-curve*, which we had under consideration in reasoning about equal-natured Xs, may thus be indicated. Take a point $\xi'_1\eta'_1$ on one of the contract-lines, say $C_1C'_1$; and let X_1 and Y_1 be placed there. Let X_2Y_2 be placed at a neighbouring point, $\xi''_1\eta''_1$, on the line $C_2C'_2$; such that (1) $\xi''_1\eta''_1$ is outside the two indifference curves drawn for X_1 and Y_1 respectively through $\xi'_1\eta'_1$; (2) $\xi'_1\eta'_1$ is outside the two indifference-curves drawn for X_2 and Y_2 respectively through $\xi''_1\eta''_1$.

FIG. 2.



Then the settlement cannot be disturbed by an X and a Y simply changing partners, rushing into each other's arms, and leaving their deserted consorts to look out for new alliances. Re-contract can now proceed only by one Y moving off with the *two* Xs, as in the previous case; by which process the system may be worked down to a neighbourhood describable as $\xi_2 y_2$. In the limit, when the number of Xs and Ys are increased indefinitely, but not necessarily equally (suppose mX, and nY, where m and n are indefinitely large); if $x_r y_r$ represent the dealings of any X, viz. X_r, and similarly ξ and η be employed for the dealings of the Ys, we should find for the $2m + 2n$ variables the following $2m + 2n$ equations:

(1) $m + n$ equations indicating that each X and each

Y is *on his individual demand-curve* (compare the condition stated below, p. 48), *e.g.*

$$x_r \frac{d\,\mathrm{F}_r(x_r y_r)}{d\,x_r} + y_r \frac{d\,\mathrm{F}_r(x_r y_r)}{d\,y_r} = 0$$

(the differentiation being of course partial).

(2) $m + n - 1$ equations indicating *uniformity* of price $\frac{y_1}{x_1} = \frac{y_2}{x_2} =$ &c. $= \frac{\eta_1}{\xi_1} = \frac{\eta_2}{\xi_2} =$ &c.

(3) A last condition, which might perhaps be called *par excellence* the equation of Demand to Supply, namely, *either* $\mathrm{S}\,x = \Sigma\,\xi$, *or* $\mathrm{S}\,y = \Sigma\eta$. Thus the dealings of each and all are completely determinate and determined.

If we transform to polar co-ordinates, we might write any individual demand-curve, as $\rho = f_r\,(\theta)$; and thence obtain two *collective demand-curves* $\rho = \mathrm{S}f\,(\theta)$ and $\rho = \Sigma\,\phi\,(\theta)$; substantially identical with those collective demand curves so scientifically developed by M. Walras, and so fruitfully applied by Mr. Marshall.

Thus, proceeding by degrees from the case of two isolated bargainers to the limiting case of a perfect market, we see how *contract is more or less indeterminate according as the field is less or more affected with the first imperfection*, limitation of numbers.

II. Let there be equal numbers of equal-natured Xs and equal-natured Ys, subject to the condition that each Y can deal at the same time with only nXs, and similarly each X with only n'Ys. First let $n = n'$. Then, in the light of the conceptions lately won, it appears that contract is as indeterminate as if the field consisted of only nXs and nYs; that is to say, there are as many and the same *final settlements* as in that case, represented by the same portion of the contract-curve

between (say) ξy and $x\eta$. Let n' increase. Contract becomes less indeterminate: ξ moving north-west, and the quantity of *final settlements* being thereby diminished. The subtracted final settlements are most favourable to the Ys. Let n' diminish. Contract becomes more indeterminate; ξ moving south-east, and the quantity of final settlements being thereby increased. The added final settlements are more favourable to the Ys than those previously existing.

The theorem admits of being extended to the general case of unequal numbers and natures.

III. Let there be an equal number N of equal-natured Xs and equal-natured Ys, and let each set be formed into equal *combinations*, there being n Xs in each X combination, and n' Ys in each Y combination. First, let $n = n'$. Then contract is *as indeterminate* as if the field consisted of $\frac{\text{N}}{n}$ Xs and $\frac{\text{N}}{n}$ Ys; in the same sense as that explained in the last paragraph. Let n' diminish. Contract becomes *less indeterminate*, in the same sense as in the last paragraph. Let n' increase. Contract becomes more indeterminate; the added final settlements being more favourable to the Ys than those previously existing.

The theorem is typical of the general case in which numbers, natures, and combinations are unequal. Combination tends to introduce or increase indeterminateness; and the final settlements thereby added are more favourable to the combiners than the (determinate or indeterminate) final settlements previously existing. Combiners *stand to gain* in this sense.

The worth of this abstract reasoning ought to be tested by comparison with the unmathematical treatment of the same subject. As far as the writer is aware,

a straightforward answer has never been offered to the abstract question, What is the effect of combinations on *contract* in an otherwise *perfect* state of competition, as here supposed? Writers either[1] ignore the abstract question altogether, confining themselves to other aspects of Trade Unionism; its tendency to promote communication, mobility, &c.; in our terms, to render the competition more *normal*, and more perfect in respect of *extent* (diminishing our *first* imperfection, for such is the effect of increased mobility, alike of goods and men). Or, while they seem to admit that unionism would have the effect of raising the *rate of wages*, they yet deny that the *total remuneration* of the operatives, the *wage-fund* (in the intelligible sense of that term), can be increased. But if our reasonings be correct, the one thing from an abstract point of view visible amidst the jumble of catallactic[2] molecules, the jostle of competitive crowds, is that those who form themselves into compact bodies by *combination* do not tend to lose, but *stand to gain* in the sense described, to gain in point of utility, which is a function not only of the (objective) remuneration, but also of the labour, and which, therefore, may increase, although the remuneration decrease; as Mr. Fawcett well sees (in respect to the question of unproductive

[1] Mr. Sidgwick indeed (if the passage already referred to, *Fortnightly Review*, p. 411, *ante*, p. 33, might be thus construed?)—at any rate some others have observed the momentous dead-lock resulting from the *complete solidification* of the whole operative-interest and the whole employer-interest; our (a) case, contract unqualified by competition. But this hardly affords any indication of what would happen, or what the writers suppose would happen, when contract is qualified, however slightly, by competition; as if, for instance, there were *two or three* combinations on one side and two or three on the other; which in view of foreign competition is likely, one might think, to be long the concrete case.

[2] Cf. Cairnes on *Trades Unions* (first sections); Courcelle-Seneuil on *Coalitions*.

consumption.—'Manual,' ch. iv.), though he gives so uncertain a sound about Trades Unionism. And if, as seems to be implied in much that has been written on this subject, it is attempted to enforce the argument against Trades Unionism by the consideration that it tends to diminish the *total national produce*, the obvious reply is that unionists, as 'economic men,' are not concerned with the *total produce*. Because the total produce is diminished, it does not[1] follow that the labourer's share is diminished (the loss may fall on the capitalist and the entrepreneur, whose compressibility has been well shown by Mr. Sidgwick in the article already referred to); much less does it follow (as aforesaid) that there should be diminished that quantity which alone the rational unionist is concerned to increase—*the labourer's utility*. If this view be correct, it would seem as if, in the matter of *unionism*, as well as in that of the predeterminate *wage-fund*, the 'untutored mind' of the workman had gone more straight to the point than economic intelligence *misled by a bad method*, reasoning without mathematics upon mathematical subjects.

IV. Let there be an equal number N of equal-natured Xs and Ys; subject to the condition that to every contract made by a Y at least n Xs must be parties, and similarly for an X n' Ys. First, let $n=n'$. Contract is as indeterminate as if the field consisted of $\frac{N}{n}$ Xs and $\frac{N}{n}$Ys. Let n' increase. Contract becomes more indeterminate, and the Ys *stand to gain*. And conversely.

To appreciate the quantity of indeterminateness likely to result in fact from these imperfections (operating separately *and together*) would require a knowledge

[1] See the remarks in Appendix VII.

of concrete phenomena to which the writer can make no claim.

The *first* imperfection applies to *Monopolies.* It is perhaps chiefly important, as supplying a clue for the solution of the other cases.

The *second* imperfection may be operative in many cases of contract for personal service. Suppose a market, consisting of an equal number of masters and servants, offering respectively wages and service; subject to the condition that no man can serve two masters, no master employ more than one man; or suppose equilibrium already established between such parties to be disturbed by any sudden influx of wealth into the hands of the masters. Then there is no *determinate*, and very generally[1] *unique*, arrangement towards which the system tends under the operation of, may we say, a law of Nature, and which would be predictable if we knew beforehand the real requirements of each, or of the average, dealer; but there are an indefinite number of arrangements *à priori* possible, towards one of which the system is urged *not* by the concurrence of innumerable (as it were) neuter atoms eliminating chance, but (abstraction being made of custom) by what has been called the Art of Bargaining—higgling dodges and designing obstinacy, and other incalculable and often disreputable accidents.

Now, if managerial work does not admit of being distributed over several establishments, of being sold in bits, it would seem that this species of indeterminateness affects the contract of an entrepreneur with foreman, of a cooperative association of workmen (or a *combination*) with a manager. This view must be modified

[1] Exceptions are the multiple intersections of Demand-Curves shown by Mr. Marshall and M. Walras.

in so far as managerial wages are determined by the *cost of production* (of a manager !), or more exactly by the equation[1] between managerial wages and the remuneration in other occupations, where the remuneration is determined by a process of the nature of *perfect* competition; and by other practical considerations.

The *third* imperfection may have any degree of importance up to the point where a whole interest (labourers or entrepreneurs) is solidified into a single competitive unit. This varying result may be tolerably well illustrated by the case of a market in which an indefinite number of consumers are supplied by varying numbers of monopolists (a case properly belonging to our *first imperfection*: namely, limited *number* of dealers). Starting with complete monopoly, we shall find the *price* continually diminish as the number of monopolists increases, until the point of complete fluidity is reached. This gradual 'extinction' of the influence of monopoly is well traced by Cournot in a discussion masterly, but limited by a particular condition, which may be called *uniformity of price, not (it is submitted) abstractedly necessary in cases of imperfect competition.*[2] Going beyond Cournot, not without trembling, the present inquiry finds that, where the field of competition is sensibly imperfect, an indefinite number of *final settlements* are possible; that in such a case *different* final settlements would be reached if the system should run down from different *initial positions* or contracts. The

[1] In virtue of *permeability* between occupations; postulating (1) freedom of choice between different occupations, (2) knowledge of circumstances determining choice. With the latter sort of knowledge (so warmly impugned by Mr. Cliff Leslie) our *free communication* about *articles of contract* (in *normal* market) is not to be confounded. See p. 18.

[2] Cf. Walras's *Elements*, s. 352.

sort of difference which exists between[1] Dutch and English auction, theoretically unimportant in *perfect competition*, does correspond to different results, *different final settlements* in imperfect competition. And in general, and in the absence of imposed conditions, the said final settlements are *not on the demand-curve, but on the contract-curve.* That is to say, there *does not necessarily exist* in the case of imperfect as there does in the case of perfect competition a certain property (which some even mathematical writers may appear to take for granted), namely, that—in the case all along supposed of Xs and Ys dealing respectively in x and y—if any X X give x in exchange for y_r, he gets no less and no more y than he is willing to take at the *rate of exchange* $\frac{y_r}{x_r}$.

If, however, this condition, though *not spontaneously generated by imperfect as by perfect competition*, should be introduced *ab extra*, imposed by custom and convenience, as no doubt would be very generally the case, nevertheless the property of *indeterminateness, plurality of final settlements*, will abide. Only the final settlements *will* now be by way of demand-curve, not contract-curve. *If*, for instance, powerful trades unions did not seek to fix the *quid pro quo*, the *amounts* of labour exchanged for wealth (which they would be quite competent to seek), but only the *rate of exchange*, it being left to each capitalist to purchase as much labour as he might demand at that rate, there would still be that sort of *indeterminateness favourable to unionists* above described. The geometry of this case may be understood from an attentive consideration of

[1] As Thornton suggests. Now we believe, but not because that unmathematical writer has told us.

the typical illustration at the end of Appendix V., fig. 4.

The *fourth* imperfection would seem likely to operate in the case of *cooperative associations* up to the time when the competitive field shall contain a practically infinite number of such bodies; that is, perhaps for a long time. To fix the ideas, suppose associations of capitalist-workmen, consisting each of 100 members, 50 contributing chiefly capital, and 50 chiefly labour. Let the *field of competition* consist of 1,000 individuals. The point here indicated is that, notwithstanding the numerical size of the field, contract will not be more determinate (owing to the fact that all the members of the association are *in contract with each other*—not, as now usual, each for himself contracting with employer) than if the field consisted of 10 individuals. And a similar result would hold if, with more generality, we suppose members contributing labour and capital in varying amounts, and remunerated for their sacrifices according to a *principle of distribution*; in the most, or, at any rate, a sufficiently general case, a *function* of the sacrifices, the form of the function being a contract-variable, or what comes to much the same thing, there being assumed a function of given form containing any number of constants, which are *articles of contract*, subject, of course, to the condition that the sum of the portions assigned is equal to the distribuend. And, similarly, if we introduce different kinds of labour and other concrete complications.

The Determinateness will depend not so much upon the number of individuals as upon the number of associations in the field. As cooperative association becomes more prevalent, no doubt, *cæteris paribus*, the indeterminateness here indicated would decrease.

Nevertheless, in consequence of the great variety of cooperative experiments, the sundry kinds of contract and divers species of *articles*, the field of competition being thus broken up, it is submitted that the rise of cooperative association is likely to be accompanied with the prevalence of[1] indeterminateness, whatever opinion we may form about the possible regularity in a distant future.

Altogether, if of two great coming institutions, trades-unionism is affected with the *third* imperfection, and cooperative association with the *fourth*, and both with the *second*, it does not seem very rash to infer, if not for the present, at least in the proximate future, a considerable extent of indeterminateness.

Of this inference what would be the consequence. To impair, it may be conjectured, the reverence paid to *competition*; in whose results—as if worked out by a play of physical forces, impersonal, impartial—economists have complacently acquiesced. Of justice and humanity there was no pretence; but there seemed to command respect the majestic neutrality of Nature. But if it should appear that the field of competition is deficient in that *continuity of fluid*,[2] that *multiety of atoms* which constitute[3] the foundations of the uniformities of Physics; if competition is found wanting, not only the regularity of law, but even the impartiality of chance—the throw of a die loaded with villainy—economics would be indeed a 'dismal science,' and the reverence for competition would be no more.

[1] There has been, I believe, observed in cooperative associations, with regard to the comparative remunerations of capital and labour, that dispute without any principle of decision which is the characteristic of contract.

[2] Above, pp. 5, 18.

[3] *Theory of Vortices and Theory of Atoms.*

There would arise a general demand for a *principle of arbitration.* . . .

And this aspiration of the commercial world would be but one breath in the universal sigh for articles of peace. For almost every species of social and political contract is affected with an indeterminateness like that which has been described; an evil which is likely to be much more felt when, with the growth of intelligence and liberty, the principle of *contract* shall have replaced both the appeal to force and the acquiescence in custom. Throughout the whole region of in a wide sense *contract*, in the general absence of a mechanism like perfect competition, the same essential indeterminateness prevails; in international, in domestic politics; between nations, classes, sexes.

The whole creation groans and yearns, desiderating a principle of arbitration, an end of strifes.

COROLLARY.—Where, then, would a world weary of strife seek a principle of arbitration? In *justice*, replies the moralist; and a long line of philosophers, from Plato to Herbert Spencer, are ready to expound the principle. But their expositions, however elevating in moral tone, and of great hortative value for those who already know their duty, are not here of much avail, where the thing sought is a definite, even quantitative, criterion of what is to be done. *Equity* and 'fairness of division' are charming in the pages[1] of Herbert Spencer, and delighted Dugald Stewart with the appearance[2] of mathematical certainty; but how would they be applicable to the distribution of a joint product between cooperators? Nor is the *equity* so often invoked by a high authority on cooperation much more available; for *why* is the particular principle of distribution recom-

[1] *Data of Ethics*, p. 164. [2] *Essays*, Book II.

mended by Mr. Holyoake (operatives to take net product, paying therefrom a salary to manager, roughly speaking, and to say nothing of capital) more equitable than an indefinite number of other principles of distribution (e.g. operatives to take *any fraction* which might have been agreed upon, manager the remainder ; *either* party, or *neither*, paying wages to the other).

Justice requires to be informed by some more definite principle, as Mill[1] and Mr. Sidgwick reason well. The star of justice affords no certain guidance—for those who have loosed from the moorings of custom—unless it reflect the rays of a superior luminary—utilitarianism.

But, even admitting a disposition in the purer wills and clearer intellects to accept the just as *finis litium*, and the useful as the definition of the just; admitting that there exists in the higher parts of human nature a tendency towards and feeling after utilitarian institutions; could we seriously suppose that these moral considerations were relevant to war and trade ; could eradicate the 'controlless core' of human selfishness, or exercise an appreciable force in comparison with the impulse of self-interest. It would have to be first shown that the interest of all is the interest of each, an illusion to which the ambiguous language of Mill, and perhaps Bentham, may have lent some countenance, but which is for ever dispelled by the masterly analysis of Mr. Sidgwick. Mr. Sidgwick acknowledges two supreme principles—Egoism and Utilitarianism ; of independent authority, conflicting dictates ; irreconcilable, unless indeed by religion.

It is far from the spirit of the philosophy of pleasure to depreciate the importance of religion ; but in the

[1] See review of Thornton on *Labour* (as well as *Utilitarianism*).

present inquiry, and dealing with the lower elements of human nature, we should have to seek a more obvious transition, a more earthy passage, from the principle of self-interest to the principle, or at least the practice, of utilitarianism.

Now, it is a circumstance of momentous interest—visible to common sense when pointed out by mathematics—that *one* of the in general indefinitely numerous *settlements*[1] between contractors is the utilitarian arrangement of the articles of contract, the contract tending to the greatest possible total utility of the contractors. In this direction, it may be conjectured, is to be sought the required principle. For the required basis of arbitration between economical contractors is evidently *some* settlement; and the utilitarian settlement may be selected, in the absence of any other principle of selection, in virtue of its moral peculiarities:

[1] Where the *contract-curve* is $\left(\frac{dP}{dx}\right)\left(\frac{d\Pi}{dy}\right) - \left(\frac{d\Pi}{dx}\right)\left(\frac{dP}{dy}\right) = 0$, the *utilitarian point* has co-ordinates determined by the equations

$$\left(\frac{d}{dx}\right)\Big[\,P + \Pi\,\Big] = 0 \quad \left(\frac{d}{dy}\right)\Big[\,P + \Pi\,\Big] = 0;$$

the roots of which evidently satisfy the contract-equation. The theorem is quite general.

Here may be the place to observe that if we suppose our contractors to be in a sensible degree *not* 'economic' agents, but actuated in effective moments by a sympathy with each other's interests (as even now in *domestic*, and one day perhaps in political, contracts), we might suppose that the object which X (whose own utility is P), tends—in a calm, effective moment—to maximise, is not P, but $P + \lambda\,\Pi$; where λ is a *coefficient of effective sympathy*. And similarly Y—not of course while rushing to self-gratification, but in those regnant moments which characterise an ethical 'method'—may propose to himself as end $\Pi + \mu\,P$. What, then, will be the contract-curve of these modified contractors? *The old contract curve between narrower limits.* In fig. 1, $y_0\ \xi_0$ will have been displaced in a north-westerly and $\eta_0\ x_0$ in a south-easterly direction. As the coefficients of sympathy increase, utilitarianism becomes more *pure*, (cf. pp. 12, 17), the *contract-curve narrows down to the utilitarian point.*

its satisfying the sympathy[1] (such as it is) of each with all, the sense of justice and utilitarian equity.[2]

These considerations might be put clearest in a particular, though still very abstract, case. Let us suppose that in consequence of combinations competition fails to determine the contract between entrepreneur and operatives. The case becomes that described under (*a*)—deadlock between two contracting parties. One of the *parties* is indeed here *collective*; but it is allowable for the sake of illustration to make abstraction of this circumstance, to abstract also the correlated bargains with capitalists, landowners, &c., and to suppose a single entrepreneur in dealing with a single operative.

And, first, let it be attempted to arbitrate upon some principle of *doctrinaire justice*—some metaphysical dogma, for instance, of equality: that the entrepreneur shall have an 'equal' share of the produce. Now, there is no presumption that this 'fair division' is utilitarianian; in view of the different character of the entrepreneur's *sacrifice*, in view also (if one may be allowed to say so) of a possible difference in the entrepreneur's *capacity*:[3] suppose, for instance, that a more highly nervous organisation required on the average a higher *minimum* of means to get up to the zero of utility. As there is no presumption that the proposed arrangement is utilitarian, so there is no presumption that it is on the contract-curve. Therefore, the self-interests of the two parties will *concur* to bulge away from the assumed position; and, bursting the cobwebs

[1] Assuming as economists assume (see Mill, book II. chap. xiv. s. 7, Walker on *Wages*, &c.), an however slight *clinamen* from the rectilinearity of the 'economic man.'

[2] Whereof the unconsciously implicit first principle is: Time-intensity units of pleasure are to be equated irrespective of persons.

[3] See p. 58.

of doctrinaire justice, to descend with irresistible force to some point upon the contract-curve. Suppose that by repeated experiences of this sort the contract-curve has been roughly ascertained—a considerable number of *final settlements* statistically tabulated. Now these positions lie in a *reverse order of desirability* for each party; and it may seem to each that as he cannot have his own way, in the absence of any definite principle of selection, he has about as good a chance of one of the arrangements as another. But, rather than resort to some process which may virtually amount to tossing up, both parties may agree to commute their chance of any of the arrangements for the certainty of one of them, which has certain distinguishing features and peculiar attractions as above described—the utilitarian arrangement.

Or perhaps, considering the whole line of possible arrangements, they might agree[1] to 'split the difference,' and meet each other in the neighbourhood of the central point—the 'quantitative mean,' as it might be called. Well, first, this quantitative mean would likely to be nearer than the extremes to the utilitarian point; and, further, this very notion of *mean* appears to be the outcome of a rudimentary 'implicit' justice, apt in a dialectical atmosphere to bloom into the 'qualitative[2] mean' of utilitarian equity.

[1] See p. 135.

[2] Aristotle's metaphysical theory that virtue is a mean between two vices is analogous to the mathematical theory that a *maximum of pleasure* is a mean between two minima.

So also Aristotle's notion of two species of excellence (ἀρετή), and more generally all cases in which there seem to be *two* (or more) *best* ways of acting (using the superlative in a sense analogous to the proper mathematical sense of 'maximum'), may be cases of *multiple solutions* of a problem in the *Calculus of Variations*, the problem of *maximum* utility.

It is difficult to allude to Mr. Todhunter's beautiful and delicate problems

Or less specifically may we say that in the neighbourhood of the contract-curve *the forces of self-interest being neutralised*, the tender power of sympathy and right would become appreciable; as the gentler forces of the magnetic field are made manifest when terrestrial magnetism, by being opposed to itself, is eliminated.

Upon the whole—omitting what it is obvious to understand about the spirit in which very abstract reasonings are to be regarded: a star affording a general direction, not a finger-post to specify a by-path—there may appear, at however great a distance, a general indication that *competition requires to be supplemented by arbitration, and the basis of arbitration between self-interested contractors is the greatest possible sum-total utility.*

Thus the *economical* leads up to the *utilitarian calculus*; the faint outlines of which, sketched in a previously published paper, may be accepted as the second subdivision of our Second Part.

UTILITARIAN CALCULUS.

PROBLEM.—To find (α) the distribution of means and (β) of labour, the (γ) quality and (δ) number of population, so that there may be the greatest possible happiness.

DEFINITIONS.—(1) *Pleasure* is used for 'preferable feeling' in general (in deference to high authority, though the general term does not appear to call up with equal facility all the particulars which are meant to be

without once more inviting attention to the versatile features and almost human complexion of that species of Calculus which seems most directly applicable to the affairs of men; so different from the brutal rigour ascribed to Mathematics by men who are acquainted only with its elements.

included under it, but rather the grosser[1] feelings than for instance the 'joy and felicity' of devotion). The term includes absence of pain. *Greatest possible happiness* is the greatest possible integral of the differential 'Number of enjoyers × duration of enjoyment × degree thereof' (*cf.* axiom below).[2]

(2) *Means* are the distributable proximate means of pleasure, chiefly wealth as destined for consumption and (what is conceivable if not usual in civilisation) the unpurchased command of unproductive labour.

(3) An individual has greater *capacity for happiness* than another, when for the same amount whatsoever of means he obtains a greater amount of pleasure, *and also* for the same increment (to the same amount) whatsoever of means a greater increment of pleasure.

This 'definition of a thing' is doubtless (like Euclid's) imperfectly realised. One imperfection is that some individuals may enjoy the advantages not for *any* amount of means, but only for values above a certain amount. This may be the case with the higher orders of evolution. Again, one individual may have the advantages in respect of one kind of means, another of another. But, if one individual has the advantages in respect of most and the greatest pleasures, he may be treated as having more capacity for pleasure in general. Thirdly, the two advantages may not go together. If 'the higher pleasures, such as those of affection and virtue, can

[1] Compare the base associations of 'Utilitarianism.' Surely, as Mr. Arnold says, a pedant invented the term.

[2] The greatest possible value of $\int\int\int dp\, dn\, dt$ (where dp corresponds to a just perceivable increment of pleasure, dn to a sentient individual, dt to an instant of time). The limits of the time-integration are 0 and ∞, the present and the indefinite future. The other limits are variable, to be determined by the Calculus of Variations.

hardly be said to come from pleasure-stuff at all' (as Mr. Barratt says in his able Note in 'Mind X.,' often cited below), it is possible (though not probable?) that the enjoyers of the higher pleasures should derive from the zero, or rather a certain minimum, of means (and *à fortiori* for all superior values) an amount of pleasure greater than another class of enjoyers, say the sensual, can obtain for any amount whatsoever of means; *while at the same time* the sensual obtain greater increments of pleasure for the same increments of means (above the minimum). In such a case the problem would be complicated, but the solution not compromised. Roughly speaking, the first advantage would dominate the theory of population; the second the distribution of means. A fourth imperfection in the statement of the definition is that the units whose capacities are compared are often *groups* of individuals, as families. With these reservations the reality of the definition may be allowed.

But it may be objected that differences of capacity, though real, are first not precisely ascertainable, and secondly artificial, being due to education. But, first, even at present we can roughly discriminate capacity for happiness. If the higher pleasures are on the whole most pleasurable—a fact of which the most scientific statement appears to have been given by Mr. Sully[1]—then those who are most apt to enjoy those pleasures tend to be most capable of happiness. And, as Mr. Barratt says, it 'seems (speaking generally) to be the fact that, the higher a being in the scale of evolution, the higher its capacity for pleasure;' while greater precision might be attainable by improved examinations and hedonimetry. Further it will be seen that some of the

[1] *Pessimism, note* to chap. xi.

applications of the problem turn upon *supposed*, rather than ascertained, differences of capacity. The second objection, William Thompson's, would hardly now be maintained in face of what is known about heredity. But it is worth observing that his conclusion, equality of distribution, follows from his premiss only in so far as a proposition like our first postulate (below) is true of wealth and labour applied to *education*, in so far as it is true that improvement is not proportionately increased by the increase of the means of education.

(4) An individual has more *capacity for work* than another,[1] when for the same amount whatsoever of work done he incurs a less amount of fatigue, *and also* for the same increment (to the same amount) whatsoever of work done a less increment of fatigue.

This fourth definition may present the same imperfections as the third. Indeed the fourth definition is but a case of the third; both stating relation between means and pleasure. The third definition becomes the fourth, if you *change the signs* of means and pleasure, put means produced for means consumed and the pains of production for the pleasures of consumption. Or not even the latter change, in so far as labour is sweet (which is very far according to Fourier). It is submitted that this identification confirms the reality of the third definition, since the reality of the fourth is undisputed. Of course, if we identify the definitions, we must bear in mind that they are liable to be separated in virtue of the second imperfection above noticed.

Axiom.—Pleasure is measurable, and all pleasures are commensurable; so much of one sort of pleasure

[1] Or this: When the same amount of fatigue corresponds to a greater amount of work done, and the same increment (to the same amount) of fatigue to a greater increment of work.

felt by one sentient being equateable to so much of other sorts of pleasure felt by other sentients.

Professor Bain has shown[1] how one may correct one's estimate of one's own pleasures upon much the same principle as the observations made with one's senses; how one may correctly estimate the pleasures of others upon the principle 'Accept identical objective marks as showing identical subjective states,' notwithstanding personal differences, as of activity or demonstrativeness. This 'moral arithmetic' is perhaps to be supplemented by a moral differential calculus, the Fechnerian method applied to pleasures in general. For Wundt has shown that sensuous pleasures may thereby be measured, and, as utilitarians hold, all pleasures are commensurable. The first principle of this method might be: Just-perceivable increments of pleasure, of all pleasures for all persons, are equateable.[2] Implicated with this principle and Bain's is the following: Equimultiples of equal pleasures are equateable; where the multiple of a pleasure signifies exactly similar pleasure (integral or differential) enjoyed by a multiple number of persons, or through a multiple time, or (time and persons being constant) a pleasure whose degree is a multiple of the degree of the given pleasure. The last expression is open to question (though see Delbœuf 'Étude psychophysique,' vii. and elsewhere), and is not here insisted upon. It suffices to postulate the practical proposition that when (agreeably to Fechnerian conceptions) it requires n times more just-perceivable increments to get up to one pleasure from zero than to get up to another, then the former pleasure enjoyed by a given number of persons during a given

[1] *Emotions and Will*, 3rd edition.

[2] Cf. Wundt, *Phys. Psych.*, p. 295; above, p. 8, Appendix III.

time is to be sought as much as the latter pleasure enjoyed by n times the given number of persons during the given time, or by the given number during the multiple time. Just so one cannot reject the practical conclusions of Probabilities, though one may object with Mr. Venn to speaking of *belief* being numerically measured. Indeed these principles of μετρητικὴ are put forward not as proof against metaphysical subtleties, but as practical; self-evident *à priori*, or by whatever ἐπαγωγὴ or ἐθισμὸς is the method of practical axioms.

Let us now approach the Problem, attacking its inquiries, separately and combined, with the aid of appropriate POSTULATES.

(*a*)[1] The *first postulate* appropriate to the first inquiry is: The rate of increase of pleasure decreases as its means increase. The postulate asserts that the second differential of pleasure with regard to means is continually negative. It does not assert that the first differential is continually positive. It is supposable (though not probable) that means increased beyond a certain point increase only pain. It is also supposable that 'the higher pleasures' do not 'come from pleasure-stuff at all,' and do not increase with it. Of course there are portions of the utilitarian whole unaffected by our adjustments; at any rate the happiness of the stellar populations. But this does not invalidate the postulate, does not prevent our managing our 'small peculiar' for the best, or asserting that in respect thereof there tends to be the greatest possible happiness. The proposition thus stated is evidenced by every-day experience; experience well focused by Buffon in his

[1] See the cumulative proofs of this postulate adduced by Professor Jevons in *Theory of Political Economy*.

'Moral Arithmetic,' Laplace in his 'Essay on Probabilities,' William Thompson in his 'Inquiry into the Distribution of Wealth,' and Mr. Sidgwick in the 'Methods of Ethics.'

This empirical generalisation may be confirmed by 'ratiocination' from simpler inductions, partly common to the followers of Fechner, and partly peculiar to Professor Delbœuf. All the formulæ suggested for the relation between quantity of stimulus and intensity of sensation agree in possessing the property under consideration; which is true then of what Professor Bain would describe, as pleasures of mere intensity; coarse pleasures indeed but the objects of much expenditure. Thus pleasure is not proportionately increased by increased glitter of furniture, nor generally by increased scale of establishment; whether in the general case by analogy from the Fechnerian experiments on the senses[1] or by a more *à priori* 'law of relation' in the sense of Wundt.

But not only is the function connecting means and pleasure such that the increase of means does not produce a proportionate increase of pleasure; but this effect is heightened by the function itself so varying (on repetition of the conditions of pleasure) that the same means produce less pleasure. The very parameter in virtue of which such functional variation occurs is exhibited by Professor Delbœuf in the case of eye-sensations;[2] that a similar variation holds good of pleasures in general is Bain's Law of Accommodation. Increase of means then, affording proportionately increased repetition of the conditions of pleasure, does not afford proportionately increased pleasure. Doubtless there

[1] Cf. Fechner, *Psychophysik*, vol. ix. p. 6.
[2] *Étude psychophysique*, &c.

are compensations for this loss; echoes of past pleasures, active habits growing up in the decay of passive impressions. Indeed the difference of individuals in respect to these compensations constitutes a large part of the difference of capacity for pleasure.

It may now be objected: increased means do not operate solely by repeating old pleasures, but also by introducing to new (*e.g.* travel); also the 'compensations' may *more than counterbalance* the accommodations. It is generally replied: In so far as a *part only* of happiness increases *only proportionately* to its means, the second differential of happiness with regard to means does not cease to be negative. That second differential cannot be *continually* negative. Its being negative for a space *may* not affect the reasoning. If it does affect the reasoning, one conclusion, the inequality of distribution, would probably (if the pleasure-curve is not very complicated) become *à fortiori*. Not only would the less capable receive then *still less* means, but even the equally capable might then not all receive equal means.

This being postulated, let us mark off the degrees of capacity for happiness on an abscissa (supposing that capacity is indicated by the values of a *single* variable; if by the values of a *function of several* variables, the proof differs only in complexity). At each degree erect an ordinate representing the number of individuals of that degree of capacity. On the rectangle corresponding to each individual it is required to construct a parallelopiped representing his means. Let us proceed to impart the distribuend means—in the first inquiry a given distribuend to given distributees doing each a given amount of labour—by way of small increments. Let us start with the assumption that each individual has

and shall retain that minimum of means just sufficient to bring him up to the zero-point of happiness (a conception facilitated by, though not quite identical with, the economical 'natural minimum of wages'). Thereafter who shall have the first increment of means? By definition an individual of the highest capacity (at least supposing the *minimum* to be the same in all capacities). Who shall have the next increment of means? *Another* individual of the highest capacity, in preference to *the same* individual by the postulate. Thus a first dividend will be assigned to the first section (all the individuals of the highest capacity) exclusively. But they will not continue sole assignees. Their means only, being continually increased, must by the postulate reach a point such that an increment of means can be more felicifically assigned to an individual of the *second section* (the next highest capacity) than to one of the first. The second section will then be taken into distribution.[1] Thus *the distribution of means as between the equally capable of pleasure is equality; and generally is such that the more capable of pleasure shall have more means and more pleasure.*

The law of unequal distribution is given by a plane curve, in the plane of the capacities and means, say a *megisthedone.* To different distribuends correspond megisthedones differing only by a *constant.* For it is educible from the postulate that there is *only one family* of megisthedones. We may have any number of *maxima* by *tacking* between different members of the family; but the *greatest possible value* is afforded by the *continuous solution.*

If we now remove the condition that each individual shall retain his minimum, what happens? Simply that

[1] Compare the reasoning in the ordinary *Theory of Rent.*

the megisthedones may now dip below the minimum line. But it is improbable that they should dip very low under the minimum at the lower end while they rise very high above the minimum at the higher end; since excessive physical privations cannot be counterbalanced by any superfluity of refined pleasures. In fact, if we assume that the zero of means corresponds to *infinite pain* of privation (*cf.* Wundt's curve of pleasure and pain), then by investigating the radius of curvature it is shown that, as the distribuend diminishes, the megisthedone tends to become a horizontal line. In famine the distribution even between unequals is equality—abstracted ulterior considerations, as of posterity.

These conclusions may be affected by the imperfections of the third definition. By the first imperfection, if the 'minimum' line were not horizontal. Secondly, suppose that the individuals who have less capacity for pleasures in general have a special capacity for particular pleasures. The bulk of means will be distributed as before, but there will be a residue distributed according to a *second megisthedone.* The second megisthedone superimposed upon the first will more or less deform it. Lastly, the unit distributee is often a *group* (*e.g.*, a married couple, in respect of their common *ménage*). The conclusions may be affected, in so far as the most capable groups are made up of individuals not most capable *as individuals.*

(*β*) The distribution of labour (to which attention has been called by Mr. Barratt) is deduced by a parity of reason from the parallel *second axiom*: that the rate of increase of fatigue increases as the work done increases, which is proved by common experience and (for muscular work) by the experiments of Professor

Delbœuf ('Étude Psychophysique'). As appears indeed from Professor Delbœuf's formulæ, the first and second postulates are to a certain extent implicated (whereby the first postulate gains strength). Let us now arrange our individuals according to their *capacity for work*, and proceed as before. Who shall do the first increment of work? Of course one of the most capable of work. And so on. *The distribution of labour as between the equally capable of work is equality, and generally is such that the most capable of work shall do more work*—so much more work,[1] as to suffer more fatigue.

The inquiry presents the same declensions as the first. In particular, cooperatives are to be compared *not inter se*, but with the *similar* operatives in similar cooperative associations: except, indeed, so far as the work done is a *symmetrical function* of the effort of fellow-workers. It is deducible that the rowers of a νηὸς ἐΐσης shall have equal fatigue; but the fatigue of the pilot is not to be equated to that of the oarsman. All the while it is to be recollected that the fatigue or *pain of work* under consideration may be *negative*.

($\alpha\beta$) To combine the first and second inquiries, determine by the Differential Calculus the constants of a *megisthedone* and a *brachistopone* such that the means distributed by the former may be equal to the work distributed by the latter *and* that the (algebraical) sum of the pleasures of consumption and the pains of production may be the greatest possible. Or, *ab initio*, by the Calculus of Variations, we may determine the *means* and *fatigue* as *independent variable functions* satisfying those two conditions.

[1] This inference requires the second form of the fourth definition, given in the Note.

$$\text{Let } V = \int_{x_0}^{x_1} n\,[F(x\,y) - p - c\{y - f\,(x\,p)\}]\,d\,x$$

where x is degree of *either* capacity, or more elegantly, if possible, a third variable in terms of which both capacities may be expressed; x_1 and x_0 are the given limits of integration (the number and quality of the distributees being not in the present inquiry variable); n is the number of each section; $F(x\,y)$ is a unit's pleasure of consumption, being a function of x his quality (capacity for pleasure) and the *independent variable* y his means; p is the unit's pain of work, another independent variable function; c is the constant incidental to problems of *relative* maximum; $f\,(x\,p)$ is the work done by the unit, being a function of his quality (capacity for work) and fatigue (effort).

Greatest possible happiness = greatest possible value

of
$$\int_{x_0}^{x_1} n\,[F(x\,y) - p]\,d\,x =$$

greatest possible value of V, c being taken so that

$$\int_{x_0}^{x_1} n\,[y - f\,(x\,p)]\,d\,x = 0.$$

The second term of the variation of V,

$$n\left[\delta\,y^2 \frac{d_2\,F}{d\,y^2} + \delta\,p^2\,\frac{d_2\,f}{d\,p^2}\right]$$

is continually negative by the postulates. Therefore the greatest possible value of V is when its first term of variation vanishes. The first term of variation,

$$n\delta y\left[\left(\frac{d\,F}{d\,y}\right) - c\right] + n\delta p\left[c\left(\frac{d\,f}{d\,p}\right) - 1\right],$$

vanishes only when both

$$\left(\frac{d\,\mathrm{F}}{d\,y}\right) = c \text{ and } \left(\frac{df}{dp}\right) = \frac{1}{c}.$$

If these equations hold, the two rules (α and β) hold. Q.E.D. The combined solution takes for granted that the means of pleasure and the pain of work *are independent* variables. And to a certain extent this may fail to be the case. An individual may want *strength* or *time* to *both* enjoy the means and do the work which the double rule assigns to him. In that case there will be a compromise between the two rules.

(γ) The *third postulate* simplifying the third inquiry is that capacity for pleasure and capacity for work generally speaking go together; that they both rise with evolution.[1] The *quality of population should be the highest possible evolution*—provided [2] that the first imperfection of the third definition does not give us pause. To advance the whole population by any the same degree of evolution is then desirable; but it is probably not the most desirable application, given quantity of a of *means of education*. For it is probable that the highest in the order of evolution are most *capable of education* and improvement. In the general advance the most advanced should advance most.

(δ) The *fourth postulate* essential to the fourth inquiry is that, as population increases, means (the distribuend) increase at a decreasing rate. This is given by the Malthusian theory with regard to the products of extractive labour. And this is [3] sufficient. For the second differential of the whole means with regard to

[1] See *New and old Methods of Ethics* (by the present writer), p. 72.

[2] *Ibid.* p. 77.

[3] This is not quite accurate. For a *part* of the distribuend may increase *more than proportionately* in virtue of economies effected by increased pro-

population is still negative, even though a *part* of means increase *proportionately* to the number of population; for instance, unproductive labour requiring little or no materials (*e.g.*, ballet-dancers), or those manufactured articles of which the cost is not appreciably affected by the cost of the raw material. From this Malthusian premiss it is deduced that *population should be limited*; but the hedonical conclusion is not necessarily of the same extent as the Malthusian (*cf.* below $\alpha\gamma\delta$). A simple inquiry under this head is the following. Assuming that all the sections (degrees of capacity or orders of evolution) multiply equally, and that each section reproduces exactly his kind, to find the (utilitarian) rate of increase?

($\gamma\delta$) A more important inquiry is: *not* assuming that all sections multiply equally, to find the average issue for each section, so that the happiness of the next generation may be the greatest possible.

First let us introduce a conception more appropriate than was possible under the preceding head; namely, that each section does not reproduce exactly its kind, but that the issue of each (supposed endogamous) section ranges on either side of the parental capacity, as thus—

$\nu = \beta\epsilon^{\frac{-(\xi-x)^2}{b^2}} \times \frac{n}{2}$; where ξ is the capacity of the parental section, n its number (= something like $A\epsilon^{\frac{-\xi^2}{a^2}}$,

duction. In the same manner, and for the same reason as a *demand-curve* may have a plurality of intersections with a vector from the origin (*Cf.* Mr. Marshall's theorem) corresponding alternately to maximum and minimum utility, so there may be a plurality of values for the sought number of population, corresponding alternately to utilitarian and pessimistic arrangement. The highest value which satisfies the equation to zero of the first term of variation must correspond to a maximum.

The imperfection of this postulate does not affect the reasoning based upon the other postulates.

since the parental generation is to be conceived as ranging under a curve of possibility; *cf.* Galton, Quetelet, &c.), ν is the number of issue of capacity x. Perhaps b is constant for all the curves of issue; the variation of β alone determines the natural maximum, or artificial limit, of the average issue. But neither the symmetry of the curves of possibility, nor the particulars of this conception, are postulated.

The *fifth postulate* appropriate to this case is that to substitute in one generation for any number of parents an equal number each superior in capacity (evolution) is beneficial for the next generation. This being granted, either analytically with the aid of Mr. Todhunter's 'Researches,'[1] or by unaided reason, it is deduced that the average issue shall be as large as possible for all sections above a determinate degree of capacity, but zero for all sections below that degree.

But can we be certain that this method of *total selection* as it might be termed holds good when we provide not only for the next generation, but for the indefinite future? In the continuous series of generations, wave propagating wave onward through all time, it is required to determine what wavelet each section of each wave shall contribute to the proximate propagated wave, so that the whole sum of light of joy which glows in the long line of waves shall be the greatest possible. If in the distant future, agreeably to the views of Herbert Spencer, population tends inartificially to become nearly stationary; if to the contemplator of all time generations fade into differentials; we may conceive formed a differential equation connecting the population of one generation with the population of its successor and in-

[1] See Appendix I. p. 93.

volving an *independent variable function*, the average issue for each section. By the Calculus of Variations (if the calculator is not at sea) it is educed that the average issue shall be as large as possible for all sections above a (for each time) determinate degree of capacity, but zero for all sections below that degree. But a further postulate is required for so long as the movement of population is not amenable to infinitesimal calculus; while the present initial irregular disturbances are far from the tranquil waves of the 'stationary' state. This *sixth postulate* might be: To substitute in one generation for any number of parents an equal number each superior in capacity (evolution) is beneficial for all time. This postulate being granted, *if possible* let the most beneficial selection be not *total*. Then a total selection can be arranged more beneficial!

If only we have swum through the waves to a *terra firma*, our position need not appear outlandish. For, first, these rules are very general, founded on very abstract tendencies, and requiring to be modified in practice. Thus our principle of selection might be modified, in so far as endogamy should not be the rule, if the higher orders of evolution have a greater tendency to reversion (in violation of the fifth and sixth postulates), and so forth. Again, since to exclude some sections from a share of domestic pleasures interferes with the principle of (α), it could not be expedient to sacrifice the present to the future, without the highest scientific certainty and political security. Again to indicate an ideal, though it can only be approached ἀνθρωπίνως, may be useful. What approach is useful in such cases is to be determined by Mr. Todhunter's principle.[1] Again, mitigations might be provided for the classes not

[1] *Researches*; below p. 93.

selected.[1] In particular, they might have the benefit of rule (β) now almost cut away by the struggle of competition. Again, *emigration* might supplement total selection; emigration from Utopia to some unprogressive country where the prospect of happiness might be comparatively zero.

($\alpha\gamma\delta$) In the preceding analysis ($\gamma\delta$) the distribution of means (and labour) was supposed given. But the reasoning is unaffected, if the distribution of means is supposed variable, provided that the later postulates are not affected by that distribution. And this they might be on Mr. Doubleday's hypothesis. But in Herbert Spencer's more probable view of the relation of affluence to populousness, the first rule (α) will become *à fortiori.*

Under this head may be considered the question: *What is the fortune of the least favoured class in the Utilitarian community?* Let us consider first the case of *emigration* for the benefit of the present generation. Let us start with the supposition, however inappropriate, that the distribuend does not vary with population; as in an isolated island where the bounty of nature could not be affected by human exertion.

The happiness of the present generation may be symbolised

$$\int_{x_0}^{x_1} n\,[\mathrm{F}(xy) - cy]\,dx + c\,\mathrm{D}$$

where D is the given distribuend and the rest of the notation is as above ($\alpha\beta$). By the third postulate x_1 is given as the highest existing degree of capacity. What remains variable is x_0, the abscissa of emigration. At

[1] *Cf.* Galton, 'The weak could find a welcome and a refuge in celibate monasteries,' &c.; also Sully, *Pessimism*, p. 392.

the limit $F(x_0 y_0) - c y_0 = 0$. Now c is positive, for it equals $\left(\frac{dF}{dy}\right)$, the first differential of pleasure with regard to means, which (presupposed a utilitarian intelligence) is probably never negative (above Postulate I.). But this is not postulated. Only, if $\left(\frac{dF}{dy}\right)$ is negative, we are dealing with the *external case* of the inquiry, determining what sections shall *immigrate* from our 'unprogressive country.' For if the Utopians have such a plethora of means that their happiness would be increased by a diminution of their means, then immigration will set in until the point of satiety be at least repassed. Then c is positive, and y is essentially positive. Therefore $F(x_0 y_0)$ is positive. It cannot be zero, the zero-point of pleasure corresponding to a positive minimum of means.

In this case *the condition of the least favoured class is positive happiness.* This conception assists us to conceive that a similar answer would be obtained if the increase of the distribuend with increasing population were *small.*

Small in relation to the megisthedonic share of the least favoured class. Write the distribuend $\int_{x_0}^{x_1} nf(xpN)dx$; where p is the effort of each unit worker, so far supposed given as a function of x; N is the number of population $= \int_{x_0}^{x_1} ndx$. Differentiate the distribuend with regard to x_0. Substitute x for x_0 and call the curve so presented the *Malthusian.* Then *the condition of the least favoured class is positive, zero,* or

negative happiness, according as at the limit the ordinate of the Malthusian is less than, equal to, or greater than that of the megisthedone.

Our uncertainty as to the condition of the lowest class increases when we consider the case of *selection* for the benefit of the next generation.

Let $n=\phi(x)$ be the curve of possibility for the present generation. Let $\nu = \mathrm{B}\epsilon / -\frac{(x-\xi)^2}{b^2} \times \frac{n}{2}$ be the curve of issue for capacity ξ; where B is the natural maximum of issue. Then n^1, the line of possibility for the next generation, is $\int_{x_0}^{x_1} \frac{1}{2}\,\mathrm{B}_{x+z}\,\epsilon^{\frac{-z^2}{b^2}} \phi(x+z)dz$, where by the fifth postulate x_1 is given as the highest existing degree of capacity; what is variable is x_0, the abscissa of total selection. The happiness of the next generation $\mathrm{H}^1 = \int_{-\infty}^{+\infty} [n^1(\mathrm{F}(x\,y)-c\,y)]dx + c\,\mathrm{D}$, where ∞ is a convenient designation for the utmost extent of *variation*—variation in the Darwinian sense. x_0 is given by the equation $\frac{d\mathrm{H}^1}{dx_0} = 0$; from which it is by no means clear that the condition of the least favoured in the second generation is above zero.

In fact, the happiness of some of the lower classes may be sacrificed to that of the higher classes. And, again, the happiness of part of the second generation may be sacrificed to that of the succeeding generations. Moreover (it is convenient, though out of order, here to add) our uncertainty increases when we suppose the laboriousness also of population variable. *Nothing indeed appears to be certain from a quite abstract point of*

view, except that the required limit is above the starving-point; both because in the neighbourhood of that point there would be no work done, and—before that consideration should come into force and above it—because the pleasures of the most favoured could not weigh much against the privations of the least favoured. (*Cf.* Wundt's pleasure-curve.)

It may be admitted, however, that a limit below the zero of happiness, even if abstractedly desirable, would not be humanly attainable; whether because discomfort in the lower classes produces political instability (Aristotle, &c.), or because only through the comfort of the lower classes can population be checked from sinking to the starving-point (Mill, &c.). Let politics and political economy fix some such limit above zero. If now Hedonics indicate a limit still superior (in point of comfort)—well. But if abstract Hedonics point to a limit *below* that hard and fast line which the consideration of human infirmity imposes, what occurs? Simply that population shall press up against that line without pressing it back.

($\beta\gamma\delta$) Under this head should be considered whether rule (β) does not interfere with rule ($\gamma\delta$). And this upon Mr. Herbert Spencer's theory of population it would do.[1] The present then may have to be sacrificed to the future; though in general how much of the present it is expedient to sacrifice to the future must be as nice a question in political, as in personal prudence.

($\alpha\beta\gamma\delta$) Contemplating the combined movements we seem to see the vast composite flexible organism, the play and the work of whose members are continually readjusted, by degrees advancing up the line of evolu-

[1] Contrast, however, Champagny, *Les Antonins*, iii. p. 277.

tion; the parts about the front advancing most, the members of the other extremity more slowly moving on and largely dying off. The final shape of the great organism, whether its bounding line of possibility shall be ultimately perpendicular, whether the graduation of (in a Greek sense) *aristocracy*, or the level of modern revolution, is the ideal of the future, is still perhaps a subject more for prejudice than judgment. Utilitarianism, indifferent about the means, with eye undistorted by prepossessions, looks only to the supreme end.

COROLLARIES. The application of these inquiries is (I.) to first principles (II.) to subordinate rules of conduct.

I. The end of conduct is argued to be Utilitarianism, as exactly defined in the 'Methods of Ethics,' by deducing from that general principle maxims of common sense; perhaps as the constitution of matter is proved by deducing from the theory experimental laws. What inferior accuracy in the moral universe indeed! But before that inferiority should prejudice, let it be settled what degree of accuracy was here to be expected. No one would listen to Professor Clerk Maxwell **πιθανολογοῦντος** about the atoms without a mathematical correspondence of his theory and the facts. But we have a large experience of the progress of Physics; it is well seen how she goes; but is the movement of Morals so familiar that the true science should be manifest by her method! Whatever the method—for Universal Eudæmonism prescribes no dogma about the origin of her supremacy; affiliated as readily to practical reason as pure passion, the 'Faith' of a Green or 'Ideals' of a Grote—whatever our faith, when we descend from faith to works, requiring a criterion for alternative actions, it may be divined that we shall not far err in

ollowing, however distantly, the procedure of the Methods of Ethics.'[1]

Consider first then Equality, the right of equals to equal advantages and burdens, that large section of distributive justice, that deep principle which continually upheaves the crust of convention.

> πολλάων πολίων κατέλυσε κάρηνα
> ἠδ' ἔτι καὶ λύσει· τοῦ γὰρ κράτος ἐστὶ μέγιστον.

All this mighty moral force is deducible from the practical principle of exact Utilitarianism combined with the simple laws of sentience (α and β).

But Equality is not the whole of distributive justice. There may be needed an ἀξία for unequal distribution. Now inequalities of fortune—abstracted the cases of governor and general and every species of trustee for the advantage of others—are generally explained by utilitarians as the consequence of conventions clear and fixed and preventing confusion and encouraging production, but not otherwise desirable, or rather of which the necessity is regretted. Yet in the minds of many good men among the moderns and the wisest of the ancients, there appears a deeper sentiment in favour of aristocratical privilege—the privilege of man above brute, of civilised above savage, of birth, of talent, and of the male sex. This sentiment of right has a ground of utilitarianism in supposed differences of *capacity*. Capacity for pleasure is a property of evolution, an essential attribute of civilisation (α). The grace of life, the charm of courtesy and courage, which once at least distinguished rank, rank not unreasonably received the

[1] Pp. 90, 346, 392, &c., 2nd edition. Cf. Buffon, *Moral Arithmetic*: 'Le sentiment n'est en général qu'un raisonnement implicite moins clair, mais souvent plus fin et toujours plus sûr que le produit direct de la raison.' (He is proving our first postulate.)

means to enjoy and to transmit (α). To lower classes was assigned the work of which they seemed most capable; the work of the higher classes being different in kind was not to be equated in severity.[1] If we suppose that capacity for pleasure is an attribute of skill and talent (α); if we consider that production is an *unsymmetrical function* of manual and scientific labour (β); we may see a reason deeper than Economics may afford for the larger pay, though often more agreeable work, of the aristocracy of skill and talent. The aristocracy of sex is similarly grounded upon the supposed superior capacity of the man for happiness, for the ἐνεργεῖαι of action and contemplation; upon the sentiment—

> Woman is the lesser man, and her passions unto mine
> Are as moonlight unto sunlight and as water unto wine.

Her supposed generally inferior capacity is supposed to be compensated by a special capacity for particular emotions, certain kinds of beauty and refinement. Agreeably to such finer sense of beauty the modern lady has received a larger share of certain *means*, certain luxuries and attentions (Def. 2; *a sub finem*). But gallantry, that 'mixed sentiment[2] which took its rise in the ancient chivalry,' has many other elements. It is explained by the polite Hume as attention to the weak,[3] and by the passionate Rousseau φυσικωτέρως.[4] Now attention to the weaker sex, and woman's right not only to certain attentions in polite society but to some exemption from the harder work of life, are agreeable to the utilitarian theory: that the stronger should not only do more work, but do so much more work as to suffer[5] more fatigue where fatigue must be suffered (β). It

[1] Cf. *Livy*, ii., p. 32, β.
[2] Burke.
[3] *Essay*, 14.
[4] *Emile*, iv.
[5] See note, p. 66.

may be objected: consideration should equally be due from the stronger to the weaker members of the same sex. But in the latter case there is wanting a natural instinct predisposing to the duties of benevolence; there has been wanting also a fixed criterion of strength to fix the associations of duty; and, lastly, competition has interfered, while competition between man and woman has been much less open (and much less obviously useful to the race). Altogether, account being taken of existing, whether true or false, opinions about the nature of woman, there appears a nice consilience between the deductions from the utilitarian principle and the disabilities and privileges which hedge round modern womanhood.

Utilitarian also is the custom of family life, among other reasons, in so far as (contrasted with communistic education) it secures for the better-born better educational influences (γ); in particular a larger share of good society in early life. The universal principle of the struggle for life, as Mr. Barratt may suggest, conduces to Utilitarian selection. This being borne in mind, there appears a general correspondence between the population-theory above deduced ($\gamma\delta$) and the current ethics of marriage, which impose[1] only a precedent condition, success, hereditary or personal, in the struggle for life. Concerning the classification of future society, common sense anticipates no utopia of equality. Physical privations are pitied; the existence of a subordinate and less fortunate class does not seem to accuse the bounty of Providence.[2] With the silence of common sense accords the uncertain sound of exact Utilitarianism ($\alpha\gamma\delta$).

But, if egoist or intuitivist are not to be altogether

[1] In respect to population.

[2] Cf. Burke on the 'labouring poor,' in *Regicide Peace*, 3.

converted by the deductive process of Mr. Sidgwick, at least the dealing with his exact definition may tend to mark out and reclaim from the indefinite one large common field of conduct, one of the virtues of the intuitivist, one of the gratifications of the egoist—rational benevolence. For can there be a rational wish to please without a willingness to estimate the duration of the pleasure, the susceptibility, as well as the number, of the pleased?

Exact Utilitarianism may also, as Mr. Barratt thinks plausible, present the end of Politics; of Politics as based upon self-interest.[1] A political 'contract' for the adjustment of conflicting interests should have two qualities. It should be clear and fixed, universally interpretable in the same sense. It should be such that the naturally more powerful class, those who, though fewer, outweigh the more numerous by strength, ability, and capacity to co-operate, should not have reason to think that they would fare better under some other contract. Two contracts present these qualities; the rough and ready *iso*cratical, the exact possibly *aristo*cratical, Utilitarianism. The first contract excels in the first quality; the second in the second.

II. That the same reasonings should lead up to a general principle and down again to its applications—that the theory should be tolerably certain, the practice indefinitely remote—is not more paradoxical than that the demonstrator of the atom-theory should foresee the remote possibility of its application, no less a possibility than to triumph over the second law of Thermodynamics.[2] The triumphs of Hedonics, if equally conceivable, are equally remote; but they do not so certainly become

[1] Compare the *Corollary* of the *Economic Calculus.*
[2] Clerk-Maxwell, *Theory of Heat*, p. 308.

more conceivable when considered more remote; for what if in the course of evolution the subtlety of science should never overtake the subtlety of feeling! Faint and vague and abstracting many things which ought not to be abstracted, the Hedonical Calculus supplies less a definite direction than a general bias, here briefly and diffidently indicated.

The end of action being defined as above, the Jacobin ideal 'All equal and rude,' J. S. Mill's ideal 'All equal and cultivated,' are not necessarily desirable, not paramount ends to be sought by revolution or the more tedious method of depopulation. Pending a scientific hedonimetry, the principle 'Every man, and every woman, to count for one,' should be very cautiously applied. In communistic association (if such should be) the distribution of produce should be rather upon the principle of Fourier than of Owen. Universal equal suffrage is less likely to be approved than plural votes conferred not only (as Mill thought) upon sagacity, but also upon capacity for happiness.

The play of the struggle for life is to be encouraged, in the present state of society, within limits, without prejudice to the supremacy of the supreme principle. Mr. Barratt indeed from the same premisses, the utility of competition, infers a different conclusion: that Utilitarianism should resign in favour of Egoism. But surely the inference is, not that the Utilitarian should change his destination from Universal to Egoistic Hedonism (points *toto cœlo* apart, as the chart of Sidgwick shows); but that, while constant to his life's star, he should *tack* (in the present state of storm at least) more considerably than the inexperienced voyager might advise. No one can misunderstand this 'self-limitation' of Utilitarianism —for it has been explained by Mr. Sidgwick; least of all

the Egoist—for a similar delegation, without abdication, of the supreme command is much more necessary in the case of the supremacy of self-love (Butler, &c.).

Lastly, while we calculate the utility of pre-utilitarian institutions, we are impressed with a view of Nature, not, as in the picture left by Mill, all bad, but a first approximation to the best. We are biassed to a more conservative caution in reform. And we may have here not only a direction, but a motive, to our end. For, as Nature is judged more good, so more potent than the great utilitarian has allowed[1] are the motives to morality which religion finds in the attributes of God.

[1] Mill, *Essays on Nature and Religion*.

APPENDICES.

I.

ON UNNUMERICAL MATHEMATICS.

It seemed undesirable to load our opening pages with a multiplicity of illustrations which, if the writer's views are correct, would be superfluous to the mathematician, and, in any case, might be uninteresting to the ἀγεωμετρητός. Indeed, the nature of the subject is such that a *single* instance—by a sort of 'mathematical induction,' as it has been called—a single 'representative-particular' authenticated instance of mathematical reasoning without numerical data is sufficient to establish the general principle. However, it may be well to add a few words of exposition after first precising the point at issue by citing on our side the father of Mathematical Economics, as the representative of the contrasted view the very able author of a review (on Prof. Jevons' 'Theory') already referred to.

Cournot says: [1]—'L'une des fonctions les plus importantes de l'analyse consiste précisement à assigner des relations déterminées entre des quantités dont les valeurs numériques, et même les formes algèbriques, sont absolument inassignables.

'D'une part, des fonctions inconnues peuvent cependant jouir de propriétés ou de caractères généraux qui sont connus, par exemple, d'être indéfiniment croissantes ou décroissantes, ou d'être périodiques, ou de n'être réelles qu'entre de certaines limites. De semblables données quelque imparfaites qu'elles paraissent, peuvent toutefois, en raison de leur généralité même, et à l'aide des signes propres à l'analyse, conduire à des relations également générales, qu'on aurait difficilement découvertes sans

[1] *Théorie des Richesses*, p. 51. See also Preface, p. viii.

ce secours. C'est ainsi que, sans connaître la loi de décroissement des forces capillaires, et en partant du seul principe que ces forces sont insensibles à des distances sensibles, les géomètres ont démontré les lois générales des phénomènes de la capillarité, lois confirmées par l'observation.'

The 'Saturday Review' (Nov. 11, 1871):— . . . 'We can tell that one pleasure is greater than another; but that does not help us. To apply the mathematical methods, pleasure must be in some way capable of numerical expression; we must be able to say, for example, that the pleasure of eating a beef-steak is to the pleasure of drinking a glass of beer as five to four. The words convey no particular meaning to us; and Mr. Jevons, instead of helping us, seems to shirk the question. We must remind him that, in order to fit a subject for mathematical inquiry, it is not sufficient to represent some of the quantities concerned by letters. If we say that G represents the confidence of Liberals in Mr. Gladstone, and D the confidence of Conservatives in Mr. Disraeli, and y the number of those parties; and infer that Mr. Gladstone's tenure of office depends upon some equation involving $\frac{dG}{dx}$ and $\frac{dD}{dy}$, we have merely wrapped up a plain statement in a mysterious collection of letters.' The reader is referred to the whole article as typical of the literary method of treating our subject. Thus, again, 'the equations . . . ., assuming them to be legitimate, seem to us to be simply useless so long as the functions are obviously indeterminable. They are merely a roundabout way of expressing what may be better said in words.' And, again, 'he wraps up his mysterious conclusions in symbols which are mere verbiage, as they contain functions which neither are nor can be determined.'

Compare Mill:—'Such principles (mathematical) are manifestly inapplicable where the causes on which any class of phenomena depend are so imperfectly accessible to our observation, that we cannot ascertain by a proper induction their numerical laws.'[1]

Compare also the spirit of his remarks[2] upon algebra and its exclusive 'adaptation to the subjects for which it is com-

[1] *Logic*, book iii. chap. xxiv. p. 9. [2] Book iv. chap. vi. p. 6.

monly employed, namely, those of which the investigations have been already reduced to the ascertainment of a relation between numbers.' Compare also the views of Comte to which he refers.

A single instance—that already cited in the text—seems sufficient to oppose to this popular impression about the limits of mathematics. Thomson and Tait, in their 'Treatise on Natural Philosophy,' p. 320, discuss the problem of a ball set in motion through a mass of incompressible fluid extending infinitely in all directions on one side of an infinite plane, and originally at rest. After constructing the Lagrangian equations from (what may be called in reference to numerical measurements) *à priori* considerations, they go on: 'principles sufficient for a practical solution of the problem of determining P and Q will be given later. In the meantime, it is obvious that each decreases as x increases. Hence the equations of motion show' several deductions which are truly 'most remarkable and very suggestive,' *e.g.* (in an analogous problem), that two balls properly projected in a perfect incompressible liquid will seem to attract one another. It is suggested, I think, that a certain hypothesis as to the ultimate constitution of matter corresponds with the observed phenomena of attraction.

Now here is the type of mathematical psychics. The 'practical solution of the problem of determining P and Q,' functions denoting quantities of pleasure in terms of external objects (means, &c.), is not yet given. But *certain properties* of such functions are given. Thus, if P be a person's pleasure considered as a function of x his means, it is obvious (compare the premises of Thomson and Tait's reasoning) that P increases as x increases, but at a decreasing rate; whence $\frac{dP}{dx}$ continually *positive*, $\frac{d_2 P}{dx^2}$ continually *negative*. And from such data mathematical reasonings show several interesting results. It has [1] been suggested that a certain hypothesis as to the ultimate principle and supreme standard of morals corresponds (to an extent not usually noticed) with the observed phenomena of human action.

[1] Above, p. 4.

One can imagine how facetious the 'Saturday Reviewer' might be in criticising the method employed by Thomson and Tait in the above example, namely, mathematical deduction without numerical measurement. As we are not able to say that P is to Q as 5 to 4, the argument 'conveys no particular meaning to us.' In employing $\frac{dP}{dx}\,\frac{dQ}{dx}$, 'we have merely wrapped up a plain statement in a mysterious collection of letters.' Doubtless, I reply, what we know of P and Q might have been stated unmathematically in a roundabout literary fashion; but that statement, as compared with Thomson's, would *not* be a *plain statement*, nor appropriate nor serviceable. For this same symbol-speech, so harsh and crabbed as compared with literary elegance, is gifted with a magical charm to win coy truth; the brief and broken language which the love of abstract truth inspires, no doubt foolishness to those who have no sympathy with that passion.[1]

What need to multiply illustrations of what is self-evident that mathematics, of which the very genius is generalisation, without dipping into particulars, soars from generality to generality! I shall attempt, however, to illustrate a little more fully the method of mathematical physics, hoping that the professed mathematician would pardon in an amateur particular errors, 'Si modo plura mihi bona sunt,' if only the general view is correct.

On the theory of sound we obtain an expression for an atmospheric wave involving two (almost) *independent arbitrary* functions,[2] $\phi\,(n\,\theta\,at - x) + \psi\,(n\,\theta\,at + x)$. *Without supposing the forms of* ϕ *and* ψ *to be known*, we may deduce sub-

[1] Here may be the place to notice the *Saturday Review's* criticism upon Professor Jevons's formulæ for the 'law of indifference:' that his symbols needlessly complicate the plain and simple facts of the market. But the most potent instruments of research are open to similar criticism. The so-called 'equation of continuity' may no doubt appear to literary common sense a very artificial and complicated statement of some such simple fact, as that matter cannot enter or leave a given space without crossing its boundary. But how fruitful of deductions is this formula in connection with other symbolic statements, needs not to tell to any one, even moderately acquainted with the kinetics of fluids.

[2] Airy on *Sound*, pp. 23, 28.

stantial conclusions; as that, when a tube is stopped at both ends, the forward and backward waves are of identical form.[1] I would not, however, insist too much on this particular instance, and the very large class of similar physical problems, as in all respects typical of psychical reasoning. For no doubt it may be said that *the data* from which the expression for wave-disturbance was deduced, the differential equation expressing the motion of a particle of air[2] $\frac{d_2X}{dt^2} = k\frac{d_2X}{dx^2}$, that this premiss is of the nature of numerical precision; k is made up of factors supposed at least approximatively measurable; whereas (some of) the data of psychics consist of *loose general relations*, the fact of increase or decrease, positive or negative, possessing not even that degree of *grossly approximative accuracy*,[3] beyond which even Professor Jevons in his illustrations of mathematical reasoning does not appear to extend his view. At the same time, if we consider as premiss the integral equation for the disturbance, then the method of psychics is fairly well exemplified by the employment in the theory of sound and elsewhere of *arbitrary functions*; a conception, one might suppose, which had never been entertained by those who object to mathematics' inability to deal with the complexities of social science; as if any degree of complexity might not be attributed to an arbitrary function.

But it would exceed the ability and requirements of the present writer to justify the method above postulated (deduction from loose and numerically indefinite relations) by a general review of the uses of arbitrary functions; it will suffice to show the validity of the method in two provinces of mathematics least distant from the sphere of psychics—I., the theory of natural forces and energy; and II., the calculus of variations.

I. The hypothesis of natural forces assumes, directly or by implication, as a first or proximate principle, that the attraction or repulsion between two particles is *some function* of the distance between them. From this loose indefinite relation, without knowledge of the form of the function, the most im-

[1] Airy, p. 78. [2] Id. p. 21. [3] *Principles of Science.*

portant conclusions may be deduced. As a very simple example take the motion of a particle round a centre of force. Without knowing the form of the force-function, we deduce that equal areas are swept out by the particle in equal times, that the motion is one plane, that the velocity is inversely proportional to perpendicular from centre upon tangent, and so forth.

No doubt it may be objected that while there is something indefinite and loose in the premisses, the hypothesis of natural forces, there is also something definite and precise, for instance, the very conception of uniform acceleration. But firstly, the hypothesis in question would generally be admitted to hold of the systems of matter immediately concomitant with mental phenomena, so that the deductions therefrom may well be of great psychophysical interest (especially in view of the analogies to be suggested between energy and pleasure). And again, it is not to be supposed that the data of social science have *nothing precise.* While there is something in them indefinite and loose, there is also something definite and precise; for instance,[1] the 'law of indifference,' that there is only one price in a market, a proposition which possesses that degree of at least approximative precision, which is generally, and supposed to be universally, characteristic of applied mathematics. And statistical data, as Professor Jevons has pointed out, admit of the same sort of precision. In fine, the objection applies at most to our *dynamical* illustrations, not to those which will be presented by pure analysis, by the calculus of variations.

The great theories relating to energy present abundantly mathematical reasoning about loose indefinite relations. Conservation of energy is implicated with such a relation, the mutual attraction of particles according to *some function* of the distance between them. The principle of conservation of energy affords instances of what is vulgarly supposed a contradiction in terms, of reasoning at once mathematically and παχυλῶς, obtaining by mathematical deduction a *general idea* of a state of motion. Suppose a swarm of particles so moving under natural forces that they are now all clustered near each other, now all fly asunder to a distance, then from the principle of the conservation of energy we obtain the *general idea* that the

[1] As aforesaid, p. 5.

movements of the particles are on an average more rapid, or more correctly their kinetic energy is greater, when they swarm together than when they are widely dispersed.

Peculiarly typical[1] of psychics are the great principles of maximum and minimum energy. That a system tends to its least potential energy, this principle affords us in innumerable instances a general idea of the system's position of rest; as in the very simple case of equilibrium being stable when the centre of gravity is as low as possible. Thus, without knowing the precise shape of a body, we may obtain a general idea of its position of equilibrium.

From the principle of least action we infer that a particle under any (natural) forces constrained to move on an equipotential surface will so move that its path from point to point is of maximum or minimum length. Without knowing the precise law of the forces, the precise shape of the potential surface, we may thus obtain a general idea of the motion.

The great Bertrand-Thompson maximum-minimum prin-

[1] The comparison between pleasure and energy may be viewed under two aspects; first (than which not more is asserted here), as not known to be more than a metaphor, yet elegant and convenient, like the hypothesis of fluids in electricity, or the 'now abandoned but still interesting' (Thomson & Tait) corpuscular theory of light; secondly, as in the text (pp. 9–15) a deep and real analogy, the *maximum* of pleasure in psychics being the effect or concomitant of a *maximum* physical energy.

The comparison assists us to conceive what appears to some inconceivable, that *equality* is not a necessary condition of greatest happiness. Energy is the product of mass and the square of velocity. Therefore the importance of any part of a system, with respect to the total energy, depends not only on its mass, but on its velocity. In the system, consisting of discharged rifle and shot bullets, there lives more energy in the little whiffling bullet than the heavy recalcitrant rifle. And, indeed, the smaller the bullet, the greater *cæteris paribus* its energy. So, in the social system, we must accustom ourselves to believe that the importance in respect to the utilitarian greatest possible quantity of each class is not necessarily in proportion to its numbers. More energy of pleasure, more ἐνεργεῖαι in the oracular language of Aristotle, may exist in one poet than many boors; in Athens than the rest of Hellas, in Hellas than Barbaria; in a century of the age of Phidias, than a thousand years of the declining Roman Empire.

No doubt this property is implicit in the definition of integral pleasure as defined, for instance, in the third Appendix. But the conception of an *integral* is not, perhaps, so familiar to the unmathematical as not to desiderate illustration.

ciples and their statical analogues present abundant instances of mathematical reasoning about loose, indefinite relations. We know, in each case, that the energy of a system to which impulses (or finite forces) have been applied is the maximum or minimum consistent with certain data. Without knowing the data precisely, we may obtain certain general ideas of the arrangement of energy in the system under consideration. Thus, if the masses of any part or parts of a material system are diminished, the connections and configuration being unaltered, the resulting kinetic energy under given (however complex and undefined) impulses from rest must be increased.[1] If the stiffness in any part or parts of the system be diminished, the connection remaining unchanged, the potential energy of deformation due to given force applied from without will be increased.[2] *Diminution* in the premisses, *increase* in the conclusion, loose, indefinite relations! So again, I think, if certain velocities be imparted by impulses to the bounding surface of an incompressible liquid, we may obtain, without having more than a general idea of the distribution of these given velocities, a general idea of the resulting motion, by reasoning, from the Thomsonian principle, that the motion of the liquid is un-rotatory, that the motion of each particle is perpendicular to a certain velocity-potential surface passing through it, one of the series of such surfaces being the bounding surface, &c. Compare with the last two paragraphs the reasonings in moral science. By first principles the arrangement (of social institutions, &c.) productive of maximum pleasure holds. Without deducing precisely *what* this best arrangement is, we may obtain mathematically a *general idea* of it as that one arrangement is better than another.

Upon analogous principles in statical electricity, we know that, if there be a given distribution of electricity over the conductors in a field, the strains throughout the dielectric are such that the potential energy of the whole system is a minimum.[3] We may not know the precise form of the functions which express the distribution of electricity over the conductors; much less, if we had these data, would we be able to

[1] Watson & Burbury, *Generalised Co-ordinates.*
[2] *Ibid.* [3] Clerk-Maxwell, *Electricity,* Arts. 98, 99.

calculate the potential, the function whose respective differentials shall give the strain in each direction at any point.[1] Yet it is something both tangible and promising to know mathematically that the potential energy is a minimum. That something is the type of what mathematical psychics have to teach. Analogous remarks are applicable to the somewhat analogous theorem of[2] minimum energy of electric currents; in a higher dimension, as I think it may be said, and of the nature of what may be called *momentum-potential* rather than force-potential.

II. It is the first principle of the calculus of variations that a varying quantity attains a maximum when the first *term of variation* vanishes, while the second term is negative (mutatis mutandis, for a minimum). The latter condition is one of those *loose, indefinite* relations which we have been all along describing. In the simple cases which in the infancy of Mathematical Psychics are alone presented in these pages,[3] we know by observation not *what* the second term is, but *that* it is continually negative. In more complicated cases the resources of mathematics are exhausted in calculating, not a definite numerical, but a loose, indefinite relation, *the sign* of the second term. The reader should consider Jacobi's method of discrimination, as stated, for instance, by Mr. Todhunter;[4] and Mr. Todhunter's application of the same to a particular problem,[5] and realise how a mathematical reasoning may turn upon the loose, indefinite relations of positive or negative, convex or concave. Consider also the many of Mr. Todhunter's 'Miscellaneous Observations' directed to the same relation. All through the calculus of variations the relation is of paramount importance, constituting, indeed, all the difference between a maximum and minimum. You find continually, in the statement of a problem,

[1] Compare Mill's or rather Comte's double objection against Mathematics in Social Science: that the premisses are unattainable, and the reasoning impossible.—*Logic*, book iv. ch. 24, p. 9.

[2] Clerk-Maxwell, Art. 283.

[3] See above, pp. 61-65.

[4] *Researches in the Calculus of Variations*, pp. 21-26.

[5] *Ibid*, pp. 26-30.

the condition that a required curve shall be, or shall not be, convex;[1] so rough and unshaped are the materials with which mathematics is able to build. Now this very relation of concavity, not a whit more indefinite in psychics than in physics, constitutes a main pillar of utilitarian calculus; quarried from such data as the law of decreasing utility, of increasing fatigue, of diminished returns to capital and labour; for the exact statement and proof of which the reader is referred to the economical writings of Professor Jevons and Principal Marshall.

It may be said that the former condition of a maximum mentioned lately, the equation of the first term of variation to zero, is of a definite precise rather than a loose indefinite character. But, again, it is to be repeated that *all* the data of mathematical psychics are not indefinite, but only (as in the case of physics) some. Accordingly, from this equation to zero, *combined with an indefinite datum*, the increase of one quantity with another, of capacity for happiness with evolution, we may deduce another indefinite quantitative relation, namely, increase,[2] or diminution of share of *means* in utilitarian distribution.

There are two other leading principles of the calculus of variations which seem calculated to illustrate the method of psychics. First, a consideration of first principles (prior, it may be observed, to any particular measurements or determination of the forms of functions), shows that if the 'Haupt Gleichung,' as Stranch calls it, the leading—in general differential—equation, which must be satisfied in order that the first term of variation should vanish, breaks up into factors, there are, or rather may be,[3] several solutions, several different functions, each corresponding to a maximum or minimum. (In the simple cases alone presented in these pages, or rather in the companion paper, in which the expression whose maximum is sought does not involve any differential co-efficients, say $\pi = \int F\,(y\,x)\,d\,x$ between limits, where y is an independent variable function; then, if $\dfrac{d\,F}{d\,y}$ breaks up into factors, there

[1] *Researches in the Calculus of Variations*, pp. 80, 117, 286.
[2] Above, p. 68. [3] Todhunter's *Researches*, p. 262.

will in general, I think, be multiple solutions.) A curve between two given points required to fulfil some maximum condition may be discontinuous, may be made up of the different solutions, one step according to one law, and the next step according to another law.[1] But the different laws or function, though they may thus be employed successively, are not to be mixed and compounded. Any one portion of the required curve must (in general and subject to the exceptions of the following paragraph), obey *some one* of the laws supplied by the solution of the *Haupt Gleichung*. It is submitted that this property has its counterpart in human affairs; the fact that there are sometimes *two best ways* of attaining an end—if the superlative *best* may be employed in a technical sense analogous to the superlative *maximum*. To realise the best, one or other course must be adopted, not a confusion of the two.

The subject of discontinuity leads up to another general remark. It is not universally necessary that the first term of variation should vanish. It suffices for a maximum that the *first term* of variation should be known to be negative (and obversely for a minimum). Such knowledge is generally the result of *imposed conditions*; as in Mr. Todhunter's problems that a curve must not pass outside a given boundary, must not exclude a given point, must be convex. It is submitted that such complicating imposed conditions have some analogy with the conditions imposed by necessity upon practical politics and applied utilitarianism. For Φρόνησις has often to be content not with the best course, but the best subject to existing conditions. Compare the subtle spirit of Mr. Todhunter's calculus of variations with the subtle, and as the 'plain man' might almost suppose, sophistical spirit of Mr. Sidgwick's method of utilitarianism, when it comes to be applied to the actual world in which we live. The abstract maximum, in psychics as well as in physics, is comparatively simple; but the concrete is complicated by imposed conditions; and the complexion of a wise benevolence, in view of each established constitution, custom, church, is affected with a congenital resemblance to the wily charms of the calculus of variations.

[1] Todhunter, *passim*.

II.

ON THE IMPORTANCE OF HEDONICAL CALCULUS.

It may be objected that mathematical psychics, though possible, are not valuable; I say *valuable* rather than, what might be understood in a too restricted sense, *useful*. For no philosophical objector would maintain that the love of the soul for the universal is then only legitimate, when it has been blessed with the production of the useful.

The love of the soul for the universal is undoubtedly capable of extravagance, as in the devotion of Plato to the idea. 'Amor ipse ordinate amandus est.' But the limits are to be traced by a loving hand, and not to be narrowed by a too severe construction of utility. The great generalisations of mathematics have perhaps been pursued and won less for the sake of utility to be produced, than for their own charm. Certainly the superior genius who reduced the general dynamical problem to the discovery of a single action-function was as much affected by the ideal beauty of 'one central idea,'[1] as by the practical consequences of his discovery. In the example first cited from Thomson and Tait, it might have happened that the generalised co-ordinates employed did not yield that 'first vindemiation' of truth above described (p. 85). Yet the Lagrangian conception of considering the energy of the whole system as a function of the position and velocities of the immersed bodies would still have been legitimate, and great, and promising. The Gossenian, the Jevonian thought of referring economics to pleasure as the central idea might be equally splendid, though unfruitful. And so Mr. G. H. Darwin, in his review of Professor Jevons's 'Political Economy,'[2] appears, not without reason, to prefer the mathematical method on theoretical, abstracted from practical, grounds.

Professor Cairnes[3] himself admits that the mathematical method might be useful, though not indispensable. If so, the

[1] Sir William R. Hamilton, *Philosophical Transactions*, 1834, 1835.
[2] *Fortnightly Review*, 1875.
[3] Preface to *Logical Method*.

position of the mathematical method in economics might be compared, perhaps, to that of quaternions, which calculus, even if it conduct to no theorem not otherwise deducible, yet, in the opinion of some [1] competent judges, deduces theorems already known more elegantly and, as it may be said, naturally and philosophically, than the blind and elephantine formulæ usually employed for the purpose. At any rate, is it for one who is not conversant with both methods to offer an opinion on their relative value; to declare forbidden, without having himself trodden, the sublimer path?

But *is* the method unfruitful in social science? The black list in our appendix may show the possibility that mathematical 'reason is here no guide, but still a guard.' But I go further, and challenge the *ἀγεωμετρητὸς* to answer the following examination paper.

Social Problems to be solved without Mathematics.

1. A communistic society owns land of varying degrees of fertility, which land it cultivates so as to obtain with a given quantity of labour the maximum of produce. Suppose the quantity of labour at the disposal of the community to be suddenly increased, how will the new labour be distributed? Will more or less additional labour be employed on any acre according as it is more or less fertile, or otherwise?

2. When Fanny Kemble visited her husband's slave plantations, she found that the same (equal) tasks were imposed on the men and women, the women accordingly, in consequence of their weakness, suffering much more fatigue. Supposing the husband to insist on a certain quantity of work being done, and to leave the *distribution* of the burden to the philanthropist, what would be the most beneficent arrangement—that the men should have the same *fatigue*,[2] or not only *more task*, but *more fatigue*?

3. Commodities being divided into two species, those whose expenses of production (do not diminish or) increase as the

[1] Cf. Tait, Edinburgh *Philosophical Transactions*, 1825.

[2] Cf. Mill's *Theory of equal sacrifice* in taxation.

amount increases and those whose cost of production diminishes with the amount produced; show that it is abstractedly expedient to tax one of these species rather than the other, and even to tax one so as to bounty the other (Marshall's theorem).

4. Commodities being divided into two species, according as a slight decrease of price is, or is not, attended with a considerable increase of demand, which species is it abstractedly preferable to tax? [1]

5. The labour market, from an indefinite number of masters and men competing on each side, is transformed by trades-unions and combinations of masters into a small number of competing (corporate) units on each side. Can this transformation be advantageous to *both* sides?

6. It has been said that the *distribution of net produce* between cooperators (labourers and capitalists associated) is arbitrary and *indeterminate*. Discuss this question.

7. Mr. Sidgwick in the 'Methods of Ethics' (iv. chap. i.), having defined the utilitarian end as the greatest possible sum of pleasures, proceeds to observe that with a view to this end *equal distribution of happiness*, though not necessarily of the *means* of happiness, is desirable. Assuming what the author's note seems to imply (cf. 'Methods of Ethics,' p. 256, 2nd edition), that individuals have their happiness differently related to means, derive *different* amounts of happiness from the *same* means; show that to attain the end defined happiness and its means must be either *both equally* or *both unequally* distributed.

There are those no doubt who see nothing in all this, turning away contemptuously from such questions, as the dog when you try to put him on a scent which nature or discipline has made to him insignificant. The professed mathematician, it must be owned with regret, is not unlikely to be in this number. But the professed mathematician, however infallible a guide upon the purely mathematical side (and sure to find many errors in these pages should they be so fortunate as to come

[1] See *Notes on Exchange Value*, by H. Cunynghame, p. 9.

under his notice) is not necessarily an infallible guide over the untrodden pass here supposed to exist between the heights of physics and psychics, supposing that his attention has not been directed to psychological problems. Nevertheless, great is the authority of the masters of the supreme science.

The authority of the mere metaphysician need give us much less pause. The noble Hegelian, from the transcendental heights whence he looks down upon Newton, might smile at the attempt to estimate quantitatively pleasure. A notable authority forsooth, this demolisher of Newton, upon the science of quantity and its limits; and notable authorities and judges of authority are those his followers, whose chosen philosopher and guide is not only blind to truth in her clearest manifestation, but also, what is even more unphilosophical, is ignorant of his ignorance and vain of his inanity. *Non ragionam di lor.* As the Olympian Zeus, defied by Here and Athena, addresses his rebuke not to the inveterately obstinate one, but only to the rebellious goddess of wisdom—

> *Ἥρῃ δ' οὔτι τόσον νεμεσίζεται οὐδὲ χολοῦται·*
> *αἰεὶ γάρ οἱ ἔωθεν ἐνικλᾶν ὅττι νοήσῃ·*

so a serious argument is addressed not to the incorrigible mystic.

Common sense is addressed and may be persuaded, it is hoped, to forego its prejudices against this sort of calculus. There is the old prejudice still reviving, however often slain, against the reign of law in psychology, as incompatible with the higher feelings. But it is too late. The reign of law is established, and will not become more oppressive to feeling by becoming mathematical. And again, common sense, catching sight of such terms as *hedonism*, is apt to dismiss the whole affair as *metaphysical*. But, it is to be insisted, the materials with which exact social science is concerned are no metaphysical shadows, but the very substance of modern civilisation, destined, doubtless, ere long to become embodied in practical politics and morals. Quantity of labour, quantity of pleasure, equality of sacrifice and enjoyment, greatest average happiness, these are no dreams of German metaphysics, but the leading thoughts of leading Englishmen and corner-stone con-

ceptions, upon which rest whole systems of Adam Smith, of Jeremy Bentham, of John Mill, and of Henry Sidgwick.

Are they not all quantitative conceptions, best treated by means of the science of quantity?

III.

ON HEDONIMETRY.

IT has been shown that some of the data of physics are as indefinite as some of the data of psychics. And yet it may be admitted that there is a potentiality of precision about even the looser physical demonstrations which gives them a certain prestige. In physics, when we deal with an indefinite P and Q (to revert to an earlier example), there is some understanding that 'principles sufficient for a practical solution of the problem of determining P and Q will be given later.' Whereas in psychics we are so far from expecting, that it seems doubtful whether we can even conceive precise measurement. Yet the conceivability at least may be thought necessary to mathematical reasoning. We must then carefully consider this possibility, or, what is much the same thing, the existence and nature of *a unit of pleasure.*

There is, no doubt, much difficulty here, and the risen science is still obscured by clouds; and hedonism may still be in the state of heat or electricity before they became exact sciences, as described by Professor Jevons.[1] Let us, however, following in his footsteps, endeavour to gain as clear a view as may be. At least it is hoped that we may sight an *argumentum ad hominem,* an argument to the man who (with Professor Jevons), admitting mathematical reasoning about self-regarding pleasures, denies the possibility of mathematically comparing different persons' pleasures. Let us accordingly, with reference to this question of μετρητικὴ and pleasure-unit consider separately the quantitative estimate which a man can form (I.) of his own pleasure, (II.) of other people's.

[1] *Theory of Political Economy,* p. 9.

I. 'Utility,' says Professor Jevons (writing exclusively of the first sort of measurement), '*may be treated as a quantity of two dimensions*.'[1] Now, when it is asked, 'In virtue of what unit is one intensity said to be greater than another?' the answer must be, I think, 'Just perceivable increments of pleasure are equatable,' which may be shown, perhaps, by that sort of internal experience and handling of ideas which seems to be the method of attaining mathematical axioms.[2] For if possible let one just perceivable increment be preferred to another. Then it must be preferred in virtue of some difference of pleasurability (non-hedonistic action not existing, or not being pertinent to the present inquiry). But, if one of the increments exceeds the other in pleasurability, then that one is not a *just perceivable* increment, but consists of at least *two* such increments. Of course such a way of turning the subject has no pretence to *de*duction. The stream of thought 'meanders level with its fount.' Turn the matter as we please, there must, I think, be postulated some such equation as the above, which may be compared, perhaps, to the first principle of probabilities,[3] according to which cases about which we are equally undecided, between which we perceive no material difference, count as equal; a principle on which we are agreed to act, but for which it might be hard to give a reason.

It must be confessed that we are here leaving the *terra firma* of physical analogy. It may plausibly be objected, the just perceivable increment, the minimum sensibile, is *not* treated as a unit in the cases with which physics deal. Let us suppose that for the same objective increase of temperature or weight (as estimated by the approved methods of physics) I have at different times, or with different organs of my body, different subjective estimates. In one sense, certainly more usual, the quantities are the same. In another sense, the minima sensibilia being equated, *what is felt is*. And this latter sense, it is contended, not without hesitation, is appropriate to our subject.

The increments in question are, I think, to be viewed as

[1] *Theory of Political Economy*, p. 51.

[2] Cf. Bain on *Axioms*.

[3] Laplace, *Essai Philosophique sur les Probabilités*, 5th edit., p. 7.

finite differences, rather than as genuine differentials (a conception which need not militate with the employment of the notation of the differential calculus).[1] The conception might be illustrated by that of a force just sufficient to turn a balance overcoming friction. Why, however, each inclination of the will is treated as equal by the rational intelligence, of this, as already intimated, no proof is to be expected.

Indeed, the equation, or equatability, in question exists not so much in fact as in the limit of perfect evolution. The imperfect intelligence does not treat a unit of pleasure in the future as equal to one in the present. Abstracting from the uncertainty of the future, the mere circumstance of futurity affects the estimate of a pleasure; which depreciation the Jevonian factor q[2] denotes, as I understand. Now it is only in the ideal limit that q becomes equal to unity.

So far about the dimension of *intensity*. As to the dimension of time a similar line of remark is open. The same objective (say horological) time may correspond to different rates of thought and feeling at different periods, as Locke intimates.[3]

It is conceivable that two states, presenting to consciousness the same number of *intensity*-increments above zero, should differ in this rate of flow. And perhaps some states, intellectual exercise in particular, which philosophers have distinguished as more good, though not more pleasurable, than others, may so differ. In dreams, the rate seems high, the intensity low. And so a pleasure would have not only two dimensions, as Professor Jevons says, but three dimensions, namely, objective time, subjective time, and intensity.

And yet the correction may not seem very important, for probably it is more competent to consciousness to combine into a single mark the two considerations of rate and intensity. Suppose one state presents about three pleasure-increments, another about two, above zero, that the rate of the former is double that of the latter, their objective duration being the

[1] See the remarks of Clerk-Maxwell, 'Essay on Atoms,' *Encyclopædia Britannica*, p. 38.

[2] *Theory of Political Economy*, p. 78.

[3] Compare *As You Like it*, Act iii. sc. 2, and elsewhere. Cf. Mr. Sully's remarks on *Illusions of Perspective*.

same, is it better to give two marks to each state, say three and two to the former, two and one to the latter, and then to multiply the marks of each; or by a sort of unconscious multiplication to mark at once six and two—*about*; for the comparison of pleasures as to quantity is here admitted to be vague; not vaguer perhaps than the comparisons made by an examiner as to excellence, where numerical marks are usefully employed.

To precise the ideas, let there be granted to the science of pleasure what is granted to the science of energy;[1] to imagine an ideally perfect instrument, a psychophysical machine, continually registering the height of pleasure experienced by an individual, exactly according to the verdict of consciousness, or rather diverging therefrom according to a *law of errors*. From moment to moment the hedonimeter varies; the delicate index now flickering with the flutter of the passions, now steadied by intellectual activity, low sunk whole hours in the neighbourhood of zero, or momentarily springing up towards infinity. The continually indicated height is registered by photographic or other frictionless apparatus upon a uniformly moving vertical plane. Then the quantity of happiness between two epochs is represented by the area contained between the zero-line, perpendiculars thereto at the points corresponding to the epochs, and the curve traced by the index; or, if the correction suggested in the last paragraph be admitted, another dimension will be required for the representation. The integration must be extended from the present to the infinitely future time to constitute the end of pure egoism.

II. Now it is here contended that there are as many, and the same sort of difficulties, in this estimate of pleasures by the sentient himself (which is yet admitted by Professor Jevons, and substantially by common sense), as in the estimate of other people's pleasures. We have only to modify our axiom thus: Any just perceivable pleasure-increment experienced by any sentient at any time has the same value. The same primal mystery of an ultimate axiom hangs, no doubt, over this utilitarian, as over the egoistic, first principle.

The equation is only true in the limit of perfect evolution.

The variation of *subjective time* for different individuals,

[1] See Clerk-Maxwell, *Theory of Heat*, p. 139.

presents no greater difficulty than the variation for one individual.

The integration may be equally well illustrated by ideal mechanism. We have only to add another dimension expressing the number of sentients, and to integrate through all time and over all sentience, to constitute the end of pure utilitarianism.

It may be objected that the just perceivable increment is given by consciousness in the case of one's own pleasures, only inferred in the case of others.[1] It may be replied, greater uncertainty of hedonimetry in the case of others' pleasures may be compensated by the greater number of measurements, a wider average; just as, according to the theory of probabilities, greater accuracy may be attained by more numerous observations with a less perfect instrument. The proposition, 'the exercise of higher intellect is accompanied with greater capacity for pleasure,' is proved by taking a wide average rather than by the self-observation, however accurate, of a single, perhaps exceptional, individual.

IV.

ON MIXED MODES OF UTILITARIANISM.

THE distinction between egoism and utilitarianism has been drawn with matchless skill by Mr. Sidgwick. But it has not been observed that between these two extremes, between the frozen pole of egoism and the tropical expanse of utilitarianism, there has been granted to imperfectly-evolved mortals an intermediate temperate region; the position of one for whom in a calm moment his neighbour's happiness as compared with his own neither counts for nothing, nor yet 'counts for one,' but counts for a fraction. We must modify the utilitarian integral as defined above (Appendix III.) by multiplying each pleasure, except the pleasures of the agent himself, by a fraction—a

[1] This is a distinction insisted on by Mr. Herbert Spencer, in his remarks on utilitarianism.—*Data of Ethics*, p. 57.

factor doubtless diminishing with what may be called the social distance between the individual agent and those of whose pleasures he takes account.

There is not much more difficulty about this intermediate conception than about the extremes. The chief difficulty is one which is common to the extremes, presented by the phenomena which Mr. Sidgwick describes as the self-limitation of a method. For example, in a life ordered according to the method of pure utilitarianism there may be tracts of egoistic action, times when the agent gives full swing to self-interest, leaving out of sight his utilitarian creed. The test whether such an agent is really a pure utilitarian would be, I suppose, whether on having his attention directed to the alternative between methods, having collected himself, in a cool moment, he would or would not calmly and deliberately sacrifice his own greatest happiness to that of others. It seems superfluous to labour a point which has been explained by Mr. Sidgwick.

Yet that there is some difficulty about this rhythm between sovereign and subordinate method may be inferred from the expressions of able thinkers. Thus, Mr. Spencer appears to employ [1] as an argument against utilitarianism the utilities of self-indulgence. 'For his wife he has smiles, and jocose speeches,' and so forth—the self-indulgent non-utilitarian. But, if self-indulgence and the not taking account of the general good has such an agreeable effect, the intelligent utilitarian will cultivate a temporary relaxation and forgetfulness of his supreme principle. It never was meant that he should wrap himself up in his utilitarian virtue so as to become a wet [2] blanket to his friends. It never was meant, as Austin says, that the sound utilitarian should have an eye to the general good while kissing his wife. In order that one's life should be subordinated to the general good, it is not necessary that the general good should be always present to consciousness. If I have an hour to prove a theorem at an examination, I shall do well not to keep the *quod est demonstrandum* continually before the mind, but to let the mind range among theorems which may serve as premisses. If a man has a day to write an article, though the whole time may be consecrated to

[1] *Data of Ethics*, chap. xi. [2] See Mr. Spencer's gloomy picture.

the purpose, it may be expedient to banish the purpose during refreshment or exercise. You cannot disprove the authority of utilitarianism by proving the utility of egoistical, or any other, practice.

To argue, then, that the utilities described by Mr. Spencer *could not* be grafted upon pure utilitarianism would imply a different conception of a 'method of ethics' from that which may be derived from Mr. Sidgwick's great work. That *as a matter of fact* the utilities of egoistic action do not now spring from a root of pure utilitarianism would be freely here admitted; agreeing with the view suggested that the concrete nineteenth century man is for the most part an impure egoist, a mixed utilitarian.

And the reconciliation between *egoism* and *altruism*, gradual process and ideal limit beautifully described by Mr. Spencer, would be upon the view suggested here, the transformation of mixed into pure utilitarianism, the psychical side of a physical change in what may be dimly discerned as a sort of hedonico[1]-magnetic field.

V.

ON PROFESSOR JEVONS'S FORMULÆ OF EXCHANGE.

PROFESSOR JEVONS'S formula, $\frac{\phi_1(a-x)}{\psi_1(y)} = \frac{y}{x}$,[2] is almost identical with our $\frac{F'_x(x,y)}{F'_y(x,y)} = \frac{y}{x}$. *Almost*; for the notation here employed is slightly more general. The utility is regarded as a function of the two variables, not the sum of two functions of each. The inquiry suggested at p. 34, near foot, could not have been suggested by Professor Jevons's formula. Our formula also is adapted to take account of the *labour of production*, the 'complicated double adjustment' glanced at by Professor Jevons.[3]

Let x manufacture the article which he exchanges for y.

[1] Above, p. 14. [2] *Theory*, p. 108. [3] *Theory*, p. 203.

Then (by a violent but not dangerous abstraction) his utility may be written

$$P = F(f(e) - x, y) - \mathcal{J}(e)$$

where e is the *objective* measure of labour (*e.g.* time of work); $\mathcal{J}(e)$ is the *subjective* measure of work, the toilsomeness of fatigue; $f(e)$ is the produce, corresponding to e. Now, as e is not an *article* of contract, it appears that $\left(\frac{dP}{de}\right)$ the partial differential with regard to e must always be equated to zero. Hence, by eliminating e we come on our old form $F(x\ y)$, or $F(-x, y)$, as it is convenient here to write.

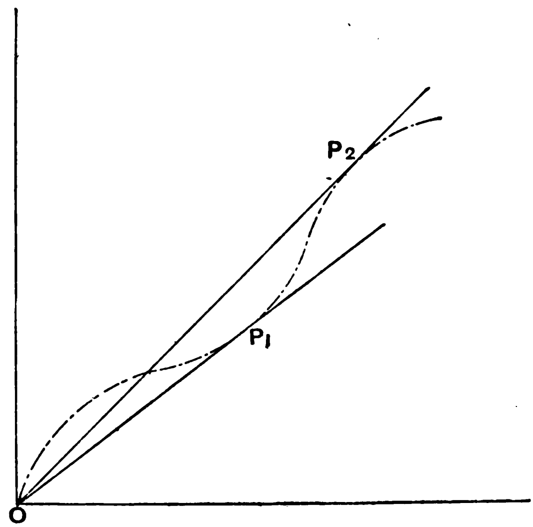
This 'complicated double adjustment' may be illustrated by a brief reference to that interesting phenomenon pointed out by both Mr. Marshall and Mr. Walras, *unstable equilibrium of trade.* From the point of view here adopted the utility of a dealer in x may be written $P = F(-x, y)$. Transformed to polar co-ordinates $P = F(-\rho \cos \theta, \rho \sin \theta)$; when $\tan \theta$ expresses the *rate of exchange.* The demand-curve is $\left(\frac{dP}{d\rho}\right) = 0$. For this locus expresses the utmost amount of dealing to which the dealer will consent at any given rate of exchange, the amount for which his utility is a maximum at that rate. But the locus *also* expresses positions for which the utility is *a minimum* at any given rate. And this part of the locus is not in a genuine demand-curve. Each point represents a position not which the dealer will not consent to change, but which he would by all means wish to change.

By a general property of analysis the maximum and minimum points are arranged *alternately* along any vector. This property is closely connected with the property of *alternately stable and unstable equilibrium of trade.* There are, however, I think, unstable positions where $\left(\frac{dP}{d\rho}\right) = 0$ does not correspond to a *minimum*, *e.g.* Mr. Marshall's figure 8.

But the most important sort of instability is perhaps that which may be presented in the case of (Mr. Marshall's) Class II; of which, as I take it, the definition connects two properties

(1) diminution of value in exchange upon increase of exports, with (2) diminution in the expense of production upon increase of wares produced for exportation. It is interesting to see from our *individualistic* point of view how these two properties are connected. The analytic condition of the first property is $\left(\frac{d_2 P}{d\rho^2}\right) = +$. For this condition must hold from the point P, where the property in question sets in (see figure) to the point

FIG. 3.



P_2, where the property ceases. At each of these points $\frac{d_2 P}{d\rho^2} = 0$. The analytic condition of the second property of Mr. Marshall's definition (the first in the order of his statement) is $\frac{d_2 f(e)}{de^2} = +$; where (as before) e is the *objective measure*[1] of labour, $f(e)$ is the amount of product corresponding to e.

It may be shown, then, that $\left(\frac{d_2 P}{d\rho^2}\right)$ can only be positive when $\frac{d_2 f}{d e^2}$ is positive. For, agreeably to previous notation, put

[1] *Other* than that which the *produce* itself presents; *e.g.*, length of time during which a uniform muscular energy is put forth by a workman.

$P = F(f(e) - \rho \cos\theta, \rho \sin\theta) - \mathcal{J}(e)$. Then we have always the condition $\left(\dfrac{dP}{de}\right) = 0$, and we have to find $\left[\dfrac{d_2P}{d\rho^2}\right]$ *subject to this condition*. Now, as θ is throughout treated as constant, whereas e is considered as a variable, dependent on ρ, it will be convenient to denote the object of our inquiry as $\dfrac{d_2P}{d\rho^2}$ *without* brackets, denoting by brackets differentiation, which is partial with respect to ρ, does not take account of e's variation. With this notation, since $\left(\dfrac{dP}{de}\right) = 0$, $\dfrac{d_2P}{d\rho^2} = \left(\dfrac{d_2P}{d\rho^2}\right) + 2\left(\dfrac{d_2P}{d\rho\, de}\right)\dfrac{de}{d\rho}$ where $\dfrac{de}{d\rho}$ is to be found from the equation to zero of

$\left(\dfrac{dP}{de}\right) = \left(\dfrac{dF}{de}\right) - \dfrac{d\mathcal{J}}{de}$. Whence $\dfrac{de}{d\rho} = -\dfrac{\left(\dfrac{d_2F}{d\rho\, de}\right)}{\left(\dfrac{d_2F}{de^2}\right) - \dfrac{d_2\mathcal{J}}{de^2}}$.

Therefore
$$\frac{d_2P}{d\rho^2} = \left(\frac{d_2F}{d\rho^2}\right) - 2\frac{\left(\dfrac{d_2F}{d\rho\, de}\right)^2}{\left(\dfrac{d_2F}{de^2}\right) - \dfrac{d_2\mathcal{J}}{de^2}}$$

Now we may be certain this expression can only be positive when $\dfrac{d_2f}{de^2}$ is positive, *if we are certain of the laws of sentience* which were postulated on a previous[1] page. For, writing a for $f(e)$ (the a employed in Professor Jevons's equation of exchange), and y for ρ sin θ, we have

$$\left(\frac{d_2F}{d\rho^2}\right) = \frac{d_2F}{da^2}\cos^2\theta + 2\,\frac{d_2F}{da\,dy}\sin\theta\cos\theta + \frac{d_2F}{dy^2}\sin\theta.$$

$$\left(\frac{d_2F}{de^2}\right) = \frac{d_2F}{da^2}\left[\frac{df}{de}\right]^2 + \frac{dF}{da}\,\frac{d_2f}{de^2}$$

where it does not seem necessary to bracket the differentials on the right-hand side. Substituting these values in the expression

[1] Page 34.

for $\frac{d_2 \mathrm{P}}{d\rho^2}$ we see that that expression is certainly negative upon these conditions:

(1) $\frac{d_2 F}{d a^2}, \frac{d_2 F}{d y^2}$ (both) continually not positive.

(2) $\frac{d_2 F}{d a\, d y}$ " "

(3) $\frac{d F}{d a}$ continually not negative.

(4) $\frac{d_2 \mathcal{J}}{d e^2}$ " "

(5) $\frac{d_2 f}{d e^2}$ not positive.

The first condition is secured by Professor Jevons's law of diminishing utility, our *first postulate* (see p. 61).

The second condition is an interesting variety of the same; that the rate of increase of utility derived from one sort of wealth diminishes with the increase of other sorts of wealth.

The third condition imports that utility at least does not decrease with increase of wealth; which in a civilised country may be allowed.

The fourth condition is Professor Jevons's law of increasing toilsomeness of labour,[1] our second axiom (see p. 65).

If then these laws of sentience hold, $\frac{d_2 \mathrm{P}}{d\rho^2}$ can only be positive when $\frac{d_2 f}{d e^2}$ is positive. It is submitted that this subordination—in however abstract and typical a form—of the more complicated phenomena of the market to the simple laws of sentience is not without interest.

But to return to Professor Jevons: the formulæ here employed, along with a general, and perhaps it ought to be added a filial, resemblance to his, present two points of contrast which deserve especial attention: (1) *Graphical illustration* has been more largely employed here. Now in some sense pure Analysis may appear to be the mother-tongue of Hedonics; which soaring above space and number deals with quantities of

[1] *Theory*, p. 185.

pleasure, employing the Calculus of Variations, the most sublime branch of analysis,[1] as Comte, Caiaphas-like, called the branch most applicable to Sociology. But on the other hand the differential equations which occur in the theory of exchange are of such a peculiar character that it is rather difficult, as may presently appear, to handle them without geometrical apparatus. In this respect at least Mr. Marshall's preference for geometrical reasoning would seem to be justified.[2]

(2) It has been prominently put forward in these pages that the Jevonian 'Law of Indifference' has place only where there is competition, and, indeed, *perfect* competition. Why, indeed, should an isolated couple exchange every portion of their respective commodities at the same rate of exchange? Or what meaning can be attached to such a law in their case? The dealing of an isolated couple would be regulated not by the theory of exchange (stated p. 31), but by the theory of simple contract (stated p. 29).

This consideration has not been brought so prominently forward in Professor Jevons's theory of exchange, but it does not seem to be lost sight of. His couple of dealers are, I take it, a sort of typical couple, clothed with the property of 'Indifference,' whose origin in an 'open market' is so lucidly described;[3] not naked abstractions like the isolated couples imagined by a De Quincey or Courcelle-Seneuil in some solitary region. Each is in Berkleian phrase a 'representative particular;' an individual dealer only is presented, but there is presupposed a class of competitors in the background. This might safely be left to the intelligence of the reader in the general case of exchange. But in dealing with exceptional cases (pp. 132, 134), a reference to first principles and the presupposition of competition would have introduced greater precision, and suggested the distinction submitted in these pages (pp. 19, &c.), namely, that exchange is indeterminate, if *either* (1) one of the trading bodies (*quâ* individual or *quâ* union) or (2) the commodity supplied by one of the dealers, be *indivisible or not perfectly divisible.*

The whole subject of the mathematical theory of exchange

[1] *Philosophie Positive*, Leçon 8. [2] *Foreign Trade*, p. 19.
[3] *Theory*, pp. 98, 99.

would be put in a clearer light by considering the objections which have been brought against Professor Jevons's theory by an able critic in the 'Saturday Review' (Nov. 11, 1871). The Reviewer says: 'When Mr. Jevons proceeds to apply this equation to the solution of his problem, he appears to us to fall into a palpable blunder. Translated into plain English, the equation $\frac{y}{x} = \frac{dy}{dx}$ means, as we see, simply that, however much corn A gives to B, he will receive a proportionate quantity of beef in exchange. If he doubles the amount of corn, that is, he will receive twice as much beef. But the other quantities are obtained on the contrary supposition, namely, that the rate of exchange will vary according to some complex law, determinable, if we could tell precisely what effect will be produced on the mind of the parties to the bargain, by the possession of varying quantities of beef and corn. In fact x is now a function of y, as might easily be foreseen from Mr. Jevons's statement of the case, in quite a different sense from what it was before. The substitution, therefore, of $\frac{y}{x}$ for $\frac{dy}{dx}$ is a mistake.'

I submit (1) the following is a significant problem. Given two differential equations $F_1\left(xy\frac{dy}{dx}\right)=0$, $F_2\left(xy\frac{dy}{dx}\right)=0$, find $\underline{x}$ and $\underline{y}$ two quantities such, that if each differential equation be solved, and thereby y for each be found as a function of x, and thence for each $\frac{dy}{dx}$ be derived as a function of x; then, if $\underline{x}$ be substituted in both (functional) values of y, and both (functional) values of $\frac{dy}{dx}$, (a) the two (quantitative) values of y are equal to each other equal to $\underline{y}$, and (b) the two (quantitative) values of $\frac{dy}{dx}$ are equal to other.

(2) The following is a solution of this problem. Eliminate $\frac{dy}{dx}$ between the equations $F_1\left(xy\frac{dy}{dx}\right)=0$, $F_2\left(xy\frac{dy}{dx}\right)=0$; the resulting equation in x and y is the locus of the required point.

(3) The problem and solution correspond to Professor Jevons's problem and solution.

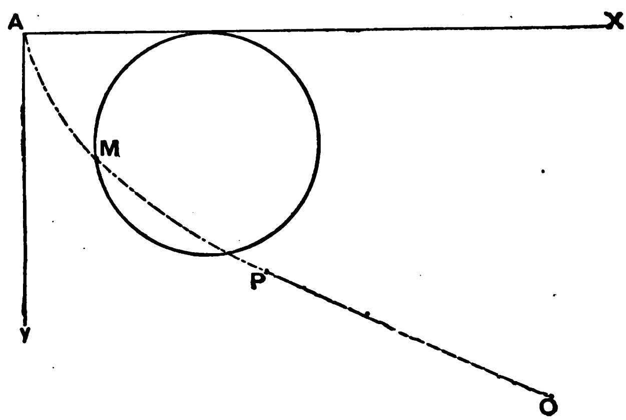
Let us take these propositions in order.

(1) This proposition by its extreme bumbledinees illustrates what was above said about the advantages of graphical illustration. For the geometrical equivalent is simply: Required a point at which two curves each given by a differential equation (of the first order) meet and touch. Or even more briefly: Find the locus of contact between members of two families.

The conception thus introduced is not only legitimate, but familiarly employed in the Calculus of Variations, in those problems where we have *multiple solution subject to the condition that there shall be no abrupt change of direction.* The reader will find any number in Mr. Todhunter's 'Researches.'

I am not concerned to show that Mr. Todhunter's problems are exactly parallel to ours. They could not well be so involving *second,* where they involve first, differentials. But it is easy to construct an exactly parallel problem with curves presented by *maximum analysis,* the source of our economical curves. Take the straight line and the cycloid, the shortest line and line

FIG. 4.



of quickest descent. A cycloid is generated by a circle of given diameter rolling on a given horizontal line, the starting-point of the circle—that is where the generating point M is on the horizontal line—being arbitrary. Find (the locus of) a point P on the cycloid such that if a particle starting from rest slide

down the cycloid from the horizontal line as far as P, and there fly off at a tangent, it will pass through a given point O.

(2) The solution above offered is easily verified. Having eliminated $\frac{dy}{dx}$ between F_1 and F_2, take any point $\underline{x}\ \underline{y}$ on the eliminant, and draw through it a curve of each family. Then $F_1(\underline{x}\ \underline{y}\, p_1)=0$; where p_1 is the value of $\frac{dy}{dx}$ for the first curve when x is substituted for x. Since the point is on the eliminant $\bar{F}_2(\underline{x}\ \underline{y}\, p_1)=0$. Also $F_2(\underline{x}\ \underline{y}\, p_2)=0$. Therefore $p_1=p_2$. Q.E.D.

In the particular case just put let the differential equation of the cycloid[1] be $\frac{dy}{dx}=\sqrt{\frac{2a-y}{y}}$, and the differential equation of the line $\frac{dy}{dx}=\frac{y-p}{x-q}$ where p and q are the co-ordinates of the given point. Then the required locus is

$$\sqrt{\frac{2a-y}{y}}=\frac{y-p}{x-q};$$

a curve of the third degree passing through the given point, as it evidently ought, *if it can*; for the given point may be too far from the horizontal line to be reached by generating circle or generated cycloid. In this last case the point is still the scene of contact between a cycloid and line, only the cycloid is *imaginary*. The mathematician is prepared for such freaks of analysis; the economist should be prepared for somewhat similar freaks[2] on the part of his similarly obtained 'demand-curve.'

To avoid misconstruction it may be as well to add that this solution by elimination of $\frac{dy}{dx}$ would *not* have been admissible if

[1] See Todhunter's *Differential Calculus*, p. 342.

[2] Thus the origin, *though an intersection of the demand-curves*, is not in any sense a position of equilibrium; not even being on the *contract-curve*. Again, the *alternate* intersections of the demand-curves are (as Messrs. Marshall & Walras have shown) positions of trade-equilibrium only in name. And we have seen that similar caution is required in handling the analytical expression of the *contract-curve* (p. 26).

there had been *other* differentials besides those of the first order. Elimination would in this case have resulted in that sort of mongrel differential equation, 'Mixtumque genus prolemque biformem,' which the Reviewer may be supposed to have had dimly in view.

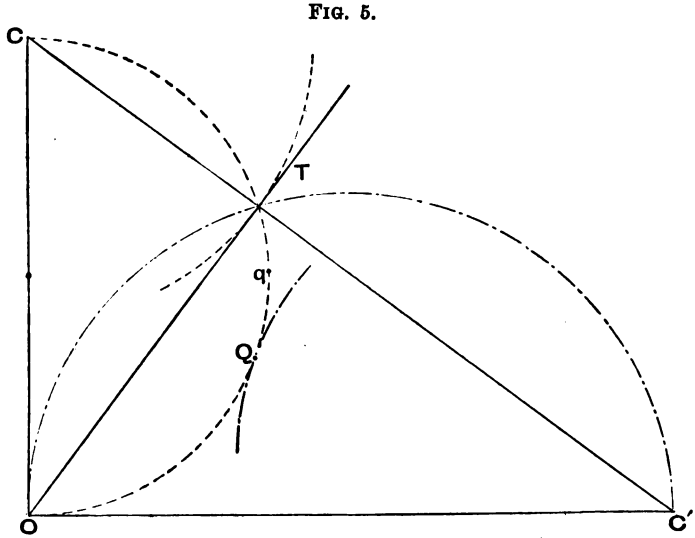
(3) An attentive consideration of Prof. Jevons's problem will show that it is a case of the problem here proposed, whether in the language of pure analysis or of geometry. I take the latter for brevity and to illustrate its convenience. Taking for origin the point at which the dealing begins where x and y are zero, we see (a) by the *law of indifference* [1] that each dealer must move along a *straight line* given by the differential equation $\frac{dy}{dx}=\frac{y}{x}$ (the Reviewer sees this much). Again under the heading 'Theory of Exchange' [2] we may learn (b) that the $\frac{dy}{dx}$ *which expresses the dealer's change of position is at the point of equilibrium* $=\frac{\phi_1(a-x)}{\psi_1(y}$. But by ($a$) the $\frac{dy}{dx}$ which expresses the dealer's change of position is *continually* $=\frac{y}{x}$. Therefore by the principles just now laid down the locus of the required point is found by eliminating $\frac{dy}{dx}$ between (a) and (b); whence $\frac{\phi_1(a-x)}{\psi_1(y)}=\frac{y}{x}$ which is none other than our old friend the '*demand-curve*.'

We may recognise another old friend in the equation $\frac{dy}{dx}=\frac{\phi_1(a-x)}{\psi_1(y)}$ considered as an ordinary differential equation. It is the differential equation of our '*curves of indifference*.' The problem under consideration may be expressed: Find the locus of the point where lines from the origin *touch* curves of indifference. If (as before supposed) the curves of indifference consist of a series of circles round a point C, then the locus of the point of contact to any curve of a tangent from O is the locus of vertices of right-angled triangles described on OC; that is, a semicircle described in OC, a result which of course might

[1] *Theory*, p. 98, *et seq.* [2] P. 103, *sqq.*

be obtained analytically according to the method here described. Transforming to the point of bisection of OC, and putting $c = \frac{1}{2}$ OC, the equation of *any* indifference-curve is $(y-c)^2+x^2=r^2$. Whence the differential equation of the family $\frac{dy}{dx} = -\frac{x}{y-c}$. And the differential equation of a straight line from O is $\frac{dy}{dx} = \frac{y+c}{x}$. Eliminating $\frac{dy}{dx}$ upon the principle here defended, we have $x^2+y^2 = c^2$ the equation of a circle whose

FIG. 5.



diameter is OC. Q E D. The determination of a point by the intersection of the locus thus obtained, with another locus similarly obtained, presents no difficulty. The conjoint determinate problem may, as we have already seen, be thus expressed. Draw from the origin a straight line, which at the same point touches two curves of indifference. As we have seen, the problem of determinate exchange may be turned in a great variety of other ways. Turn it as you will, the essential correctness of the formula under consideration emerges clearer.

Merses profundo; pulchrior evenit.
Luctere; multâ proruet integrum
Cum laude victorem.

The remaining objections of the Saturday Reviewer against this formula are based upon the interpretation already shown to be erroneous that the formula is applied to solitary couples, such as those which political economists delight to place in lonely islands. It happens, indeed, that the Reviewer is not enabled by his literary method to deduce correct conclusions from these premisses of his own assumption.[1] But we are here concerned not with his fallacious reasoning from assumed premisses, but with his undue assumption of premisses or ignorantia elenchi. We are only concerned to show that his objection does not apply to a *typical* couple *in a market.*

He puts the case of A and B, dealing respectively in corn and beef, and supposes that at a certain rate 5 of corn to 1 of beef A would exchange 20 of corn against 4 of beef and no more. Now, in so far as this objection might apply to the *typical* formula which we have been building—I do not say that the Reviewer aimed at this structure, but I am concerned to show that he does not hit it—it might import that a typical dealer would refuse to deal if the price of his article were to be raised, would not consent to such a rise of price, which surely requires no refutation. In symbols, P being the utility of dealer in x, and $\tan\theta$ the rate of exchange, $\frac{dP}{d\theta}$ is continually $+$; it being understood, of course, that movement *is along the demand-curve* of P; for, as we are here concerned with *typical* individuals *in a market*, there is no talk of movement other than along demand-curves, and the case put shows that the position of the index is on P's demand-curve, say at the point q (on the last figure).

Well, then, subject to this condition, namely $\left(\frac{dP}{d\rho}\right) = 0$, P increases continually with θ. For

$$\frac{dP}{d\theta} = \left(\frac{dP}{d\theta}\right) + \left(\frac{dP}{d\rho}\right)\frac{d\rho}{d\theta} = \left(\frac{dP}{d\theta}\right) = \frac{dF}{da}\sin\theta + \frac{dF}{dy}\cos\theta$$

[P being here supposed $= F(a - \rho\cos\theta, \rho\sin\theta)$], which is

[1] An attentive consideration of his hypothesis will show that he supposes that there can be a *settlement not on the contract-curve*; which is untenable.

continually + , unless it can be supposed that wealth can so increase as to become a *disutility*. Q. E. D.

But, it may be said, and not without plausibility, of course A would be willing enough to make the change you describe, but B, though by hypothesis he is willing to make changes in *some* direction, is not willing to make a change in *that* direction. And, true enough, a mere B, unclothed with the properties of a market, might well be unwilling to make *that* change. Referring to the same figure, let us suppose that B's *curves of indifference* are circles with C′ at centre. Then we see that for all points *above* Q where a *curve of indifference* of B touches the *demand-curve* of A, it will not be for the interest of the individual B to move up the demand-curve of A. But the *typical competitive representative* B cannot help himself. The force which moves him is not his maximum utility *barely*, but subject to competition; the best that he can get in short. And this play of the market, as fully explained here and by other writers, leads to the formulæ[1] which have been so often returned to our inquiry.

VI.

ON THE ERRORS OF THE ἀγεωμέτρητοί.

'ECQUID tu magnum reprendes Homerum,' 'Egregio in corpore nævos,' and whatever adage is applicable to carping smallness, might occur at sight of the undermentioned names, if the critic did not hasten to disclaim any disrespect for these great names, and to explain that the argument of this work, to

[1] If, however, the competition between the Bs is *not perfect*, it may happen that they cannot force each other up to T, the intersection of the demand curves; but that the system will reach a *final settlement* at some intermediate point q (as intimated at p. 48), supposing that the system is *constrained* to move along the demand-curve of A (our old X); for in the absence of this *imposed condition* it would run down to a final settlement on the contract-curve, not necessarily nor even probably T, the point where the demand curve intersects the contract-curve (in this case a straight line), CC′.

vindicate the mathematical method in Social Science, could only, or would best, be completed by showing that the profoundest thinkers would have thought more clearly upon Social Science if they had availed themselves of the aid of Mathematics.

And, if after all it appears to the reader that the list of the accused and that the accusation are not of very formidable length, he will please to consider—with reference, at least, to the two first and the two last of the reviewed authors—both *who* they are who are here suspected to have erred, and *what* the subject of their error. If *these* have erred from want of mathematical aid, what shall we expect from the unaided reason of others? And, if there is obscurity about the conception of the *ends* of action, must there not be error and confusion about the means—about all the middle axioms of morality? 'If the light that is in thee be darkness, how great is that darkness.'

BENTHAM.

That the great[1] Bentham should have adopted as the creed of his life and watchword of his party an expression which is meant to be quantitatively precise, and yet when scientifically analysed may appear almost unmeaning, is significant of the importance to be attached to the science of quantity. 'Greatest happiness of the greatest number'—is this more intelligible than 'greatest illumination with the greatest number of lamps'? Suppose a greater illumination attainable with a smaller number of lamps (supplied with more material), does the

[1] I am aware that Bentham is said by Bowring (*Deontology*, p. 328) to have corrected this phrase in later life. It was not, however, corrected in his latest works (*Constitutional Code*, chs. ii. vii.). And at any rate, as our contention is not for victory but for the sake of instruction, *οὐ περὶ τρίποδος 'Αλλὰ περὶ ψυχῆς*, it may be useful to note the errors of genius, even if they were at length self-corrected.

If after the preceding, and in view of a subsequent (p. 130), admission, the criticism in the text appears hypercritical, let it be applied only to such of Bentham's followers as may have been led by Bentham's incautious use of the phrase (*e.g. Fallacies of Confusion*, ch. iii. f. 2) into exaggerating the democratic or isocratic tendencies implicit in *Utilitarianism*; to Bentham's predecessors also, Priestley, and Beccaria, with his 'La massima felicità divisa nel maggior numero.'

criterion in this case give a certain sound? Nor can it be contended that variation of number could not have been contemplated in Bentham's day. For, supposing the number of distributees fixed, and as before a fixed distribuend, might not the sum-total of happiness be greatest when the greatest part of the sum-total, or at any rate larger portions, were held by a few? Which perhaps the aristocratic party, if they would express themselves precisely, might contend.

The principle of greatest happiness may have gained its popularity, but it lost its meaning, by the addition '*of the greatest number*.'[1]

J. S. Mill.

Nor is Mill any clearer about the definition of the Utilitarian End; indeed, darkens the subject (as many critics seem to have felt), by imposing the condition of equality of distribution. Suppose that 'equality of sacrifice,' which he lays down as the principle of taxation, should not correspond to 'least possible sum-total of sacrifice,' what then?

In the Political Economy of Mill occur some fallacies of the species under notice, on which it is unnecessary to dwell, since they have been more than abundantly exposed by Professors Jevons and Walras.[2] It might be possible, indeed, to maintain that these critics have been unnecessarily severe, and that the tone of Mr. Marshall improving upon Mill by the aid of Mathematics is more proper.[3] Thus Mill's definition of Value appears to be the same, though not always, perhaps, so well expressed, as that of Professor Jevons. And again, it might be possible for Mill to have a saving knowledge of the mysteries of Supply and Demand, even though he may have acknowledged, not two equations, but one equation.[4] For it is possible mathematically to subsume several equations in one condition. Thus the equation to zero of Virtual Velocities includes in the general

[1] See this point examined in *New and Old Methods of Ethics*, by the present writer.

[2] *Theory of Political Economy*, 2nd edition; *Eléments d'Economie Politique.*

[3] *Theory of Pure Trade*, ch. i. pp. 4, 12.

[4] *Theory of Political Economy.*

case of a free rigid body *six*, and may include any number of equations. And thus we have seen reason to suppose that all the equations of Political Economy, however numerous, may be subsumed under one.[1] And, to come nearer the mark, we have seen above that the conditions of trade-equilibrium are not necessarily stated in a bilateral and symmetrical form, but may be subsumed in a *single solitary* condition, the *equation of Demand to Supply*; presupposed and understood—what, in fact, economists only too readily[2] presuppose and take for granted—two sets of conditions, which might be described as (1) the *fact*, (2) the *uniformity* of price.[3]

But it is none of our part, Agamemnon-like, 'through the camp to go and rob an ally,' rather than 'despoil a foe.'[4]

If an author will use unmathematical language about mathematical subjects, he must expect a doubtful interpretation and fame.

Professor Cairnes.

Professor Cairnes's substantial contributions to the matter of Political Economy might surely have been enhanced by being framed in a more mathematical form.

It will be found very difficult to seize the connotation of the phrase 'increase in the aggregate amount of values.'[5] The denotation, the two instances immediately preceding, does not appear to afford any significant common attribute to constitute a definition.

The amazing[6] blindness of this author in view of the mathematical theory of exchange, his inability to contemplate scientifically the psychical mechanism underlying the phenomenon[7] of exchange, must vitiate, one should think, what he

[1] Mr. Walras has discerned the all-comprehensive character of the principle of Maximum (*Eléments*, Leçon 15); though he has not ventured, as far as I am aware, to identify Hedonical with Physical Maximum.

[2] If our reasonings are right. See Index *sub voce* Price.

[3] Above, p. 42.

[4] Pope, *Iliad*, i.

[5] *Leading Principles*, p. 5.

[6] See the only too lenient criticism of Mr. Geo. Darwin in *Fortnightly Review*, 1875.

[7] *Ibid.* p. 15.

has to tell us of 'demand' in ponderous phrase, or of 'supply, as the desire for general purchasing power.' . . . This is a subject as to which he who despises the science of quantity is not likely, as Plato would say, to be himself *ἐνάριθμος*.

No doubt he occasionally detects a vulnerable point in Mill (p. 116) which had already been more clearly exhibited by Professor Jevons. Still I venture to think that the contentions of Professor Cairnes about the definition of Supply and Demand are much more a dispute about words than could be evident to one who had no grasp of the forces determining a market. Let the facts, with sufficient accuracy for the present purpose, be summed up in Professor Jevons's symbolic statement, [1]

$$\frac{\phi_1(a-x)}{\psi_1(y)} = \frac{y}{x} = \frac{\phi_2(x)}{\psi_2(b-y)},$$

where ϕ ψ are the first differentials of Φ Ψ, and *e.g.*, $\Psi_1(y)$ represents the utility to dealer No. 1 of the quantity y of commodity No. 2; in the simplest abstract case the pleasure to be at once obtained by the consumption of y, but in the general case the pleasure to be obtained both in the immediate and more distant future, reduced to the common measure so to speak of present pleasure (by way of the Jevonian factors for *risk* and *remoteness*),[2] the pleasure I say to be thus obtained from *having now* the quantity of y (whether to be consumed gradually or perhaps exchanged for other commodities).

When the fact expressed by the symbolic statement has been grasped, it is only a dispute about words, whether we define

(1) Supply of commodity No. I. $= a$.
Supply of commodity No. II. $= b$.[3]

(2) Demand of commodity No. I. at rate of exchange $\left(\frac{y}{x}\right) = x$ (the usual definition, I think).

Demand of commodity No. II. at rate of exchange $\left(\frac{x}{y}\right) = y$.

[1] *Theory*, p. 108. [2] *Theory*, pp. 36, 38.
[3] Cf. Cairnes, p. 117.

(3) Demand for commodity No. I. is measured by the quantity y exchanged for x.[1] (?)

(4) Demand is *the desire* for commodities, &c.[2] Such language is justified, though it is not pretended that Cairnes uses it with any definite meaning, by the *first intention* of the term 'demand.'[3] In this case the demand for y might perhaps be represented by $\psi(y)$.

But I know what angry susceptibilities are awakened by the dogmatic terms Supply and Demand, and decline a contest in a region which has been darkened by such clouds of dust.

Professor Cairnes's whole contention that 'cost means sacrifice,' &c. (p. 60), may seem an unconscious tribute to the importance of the quantification and measurement of the sense of sacrifice, subjective labour. If it is admitted that on the whole he uses his 'sacrifice' and 'cost of production'[4] as an *objective* not a *subjective* quantity, 'cost as measured in number of days, labour, and abstinence' (p. 389), our e rather than our $\int(e)$;[5] still he may seem both to have had the latter quantity in view, and to have foregone some of the advantages which would have been obtained by more clearly distinguishing it.

Professor Cairnes's exposition of the bargain between employer and employed would probably have been enhanced by the use of demand-curves, one representing the quantity of work which the labourer is willing to give, and the other the (total) amount of remuneration which the employer is willing to give, at a certain rate of wages. It would have been suggested that the Wage-Fund or -Offer, though for a given rate of wages it have a *determinate*, has not necessarily a *unique*, value. The demand-curves may intersect more than once. It would not then, I think, be inconsistent with the premisses, though it might be with the conclusions, of Cairnes, that the effect of a trades-union might be to shift the position of the bargain from the first to the third (or rather from third to first) intersection. Also it would have been suggested as above, that, though the labourer might have less total remuneration in consequence of

[1] Cairnes, pp. 24, 25. [2] Id. p. 21.
[3] Cf. Cunyngham, *Notes on Exchange Value*, p. 1.
[4] Cf. 62, 63, 79, &c. [5] Appendix IV.

a trades-union, yet he might have more utility, having less labour.

Mr. Spencer.

Mr. Spencer has 'tried' the Utilitarianism of Mr. Sidgwick ('Data of Ethics'), and condemned it; but had the procedure been according to the forms of quantitative science the verdict might have been different. 'Everybody to count for one' is objected to Utilitarianism,[1] but this equation as interpreted by Mr. Spencer does not enter into Mr. Sidgwick's definition of the Utilitarian End, greatest possible product of number × average happiness,[2] the definition symbolised above.[3] Equality of distinction is no *proprium* of this definition; *au contraire*.[4] Not 'everybody to count for one,' but 'every just perceivable increment of pleasure to count for one,' or some such definition of the pleasure unit,[5] is the utilitarian principle of distribution.

(S. 85.) The case of A B, C D, producers, among whom the produce is to be distributed, presents no theoretical difficulty to the 'impartial spectator,' armed with the Calculus of Variations. The most *capable of work* shall do most work; the most *capable of pleasure* shall have most produce.[6] How could the principle of equity be worked in the entangled case of co-operative work?[7] But to the principle of greatest happiness all is simple. Consider the whole produce as a given function of the fatigues of the labourers, the pleasure of each as a given function of his portion; and determine the fatigues and the portions so that the sum of the pleasures, *minus* the sum of the fatigues, should be the greatest possible, while the sum of the portions equals the whole produce.[8]

(S. 86.) To insist that altruistic requires egoistic pleasure, is open to the remarks above made (Appendix IV.). As to the physical illustration (p. 228), grant that, in order that the whole may be heated, the parts must be heated. What then? Is it not conceivable that to each part should be imparted *just*

[1] *Data of Ethics*, ch. xiii.

[2] Book iv. ch. 1, § 2.

[3] See above, p. 57.

[4] See Index *sub voce* Equality.

[5] See above, p. 8.

[6] See above.

[7] See above, p. 51.

[8] See above, p. 64–67.

that amount of heat which may conduce to an *integral maximum*. The illustration suggests a very different view from the author's, viz., that there should *not* be 'equalness of treatment.' Let us state, as the end to be realised, that the average temperature of the entire cluster, multiplied by the number of the elements, should be the greatest possible. Let us suppose that the elements have different *thermal capacities*, or that the same amount of energy being imparted causes different increases of temperature; and (not troubling ourselves about the conservation of energy) that each element, without diminishing its own temperature, increases by radiation the temperature of its neighbours. If thermal capacity (the received definition of the term being *inverted* for the sake of the metaphor)[1] and power of radiation and absorption go together,[2] then the larger portions of a given fund of energy shall be assigned to higher capacities.

The possibility of differences of capacity in the final state of equilibrium does not seem to be entertained by the author. But can we receive this? Can we suppose that the Examination-list of the Future will consist of an all-comprehensive bracket? If capacities for work differ, possibly also capacities for pleasure.[3] If either or both species continue to differ, Utilitarianism, it is submitted, will continue to have a function not contemplated by the Data, unequal distribution.

A general agreement has been already[4] expressed with the author's view that Pure Utilitarianism is not now absolutely right. Some comment, however, may be made upon the suggested comparison between 'absolute' rightness in the case of an irregular imperfectly evolved society and mathematical certainty in the case of 'crooked lines and broken-backed curves.' Take a piece of string as crooked and broken-backed as you please, and impart to its extremities given impulses. Then it is mathematically deducible and accurately true[5] that

[1] See Clerk-Maxwell, *Heat*, p. 65.

[2] Capacity for self-regarding and for sympathetic pleasures, each probably increasing with evolution. [3] See above, p. 59, and below, p. 131.

[4] Appendix IV.

[5] Bertrand's *Theorem*, Thomson & Tait. Cf. Watson & Burbury, *Generalised Co-ordinates*, Arts. 16, 17.

the initial motion of each element is such that the whole initial energy of the string shall be maximum. No doubt to actually determine by the Calculus of Variations the motion for each element, we must know the (original) form of the string. If that form is broken-backed, a definite curve may be hypothetically assumed. So then it *might be* even now absolutely right that each individual should act so that the general happiness, as defined by Pure Utilitarianism, should be a maximum; though what that action is can only be approximately determined.

MR. SIDGWICK.

Mr. Sidgwick's Economical reasonings have been already noticed. Close and powerful as these reasonings are, it has been impossible to conceal the impression that this distinguished analyst would have taken the field in Economical speculation in a manner more worthy of himself if he had not embraced the unfortunate opinions of Cairnes [1] upon the application of Mathematics to Political Economy.

Probably the only flaws in Mr. Sidgwick's ethical analysis are where mathematical safeguards were required.

In the 'Methods of Ethics,' [2] after defining the Utilitarian End as the greatest sum of happiness, he supposes (as I understand, but it is always very difficult to catch hold of those who use ordinary language about mathematical subjects) that *happiness*, though not the *means of happiness*, should be distributed equally. But this supposition is repugnant to his definition. For, in general, either the capacities for happiness (as defined above, p. 57) are, or are not, equal. If they are equal, then both happiness and means should be distributed equally; if unequal, neither (p. 64). The supposition, then, that *happiness*, though not the means of happiness, should be distributed equally, is in general repugnant to the Utilitarian End.

[1] *Fortnightly Review*, February, 1879, p. 310. It is not for one whose views about changes in the 'general purchasing power of gold' are very hazy to criticise a theory of that subject. It may be allowable, however, to mention that the haze has not been removed by the theory of 'aggregate price,' &c., advanced in the article cited.

[2] Book iv. p. 385.

In general; for the beauty of mathematical analysis [1] is that it directs our attention not only to general rules but to exceptions. Suppose the two properties which constitute the definition of *capacity for happiness* not to go together, as in the third imperfection of that definition noticed on the same page; then it is just possible that a given distribuend would be most felicifically distributed among given distributees when the *happiness*, though not the means of happiness, should be distributed equally.

The interpretation that Mr. Sidgwick, in the passage just discussed, has in view differences of *capacity for happiness*, is confirmed by explicit recognition of such (p. 256), 'Some require more and some less to be equally happy.' The problem raised in that context is not treated with mathematical precision. 'We should have to give less to cheerful, contented, self-sacrificing people, than to the selfish, discontented, and grasping, as the former can be made happy with less.' The case would seem to be this: the minimum of means corresponding to the zero of happiness (above, p. 64) is higher for the discontented than the cheerful; for values of means above that minimum the cheerful have greater capacity for happiness. If, then, the distribuend be sufficient to admit of all at least reaching the zero of happiness, then the cheerful shall have a larger portion of means. (See above, pp. 57, 65.)

These are slight steps of reasoning; but they are at an enormous height of generalisation, where a slip is ruin.

[1] I cannot refrain from illustrating this proposition by one more reference to Principal Marshall's and Professor Walras's similar—doubtless independent—theory of multiple intersection of demand-curve, unstable equilibrium of trade.

VII.

ON THE PRESENT CRISIS IN IRELAND.

THE consideration, however superficial, of a real case may serve to put our method in a clearer light. Let us suppose, then, that an intelligent reader, attracted by the heading of this Appendix, inquires of what possible use can Psychical Mathematics be in real life?

First, it must be pointed out that deductive reasoning is not to be too sharply pulled up with the demand, 'What then do you propose?' For, even if this highly deductive method should prove more potent than the present tentative sketch may warrant, it would have power only to give general instructions, not detailed regulations. From such a height of speculation it might be possible to discern the outlines of a distant country, but hardly the by-paths in the plain immediately below. Mathematical Psychics would at best furnish a sort of pattern-idea to be roughly copied into human affairs;[1] in the language of modern Logic hypothetical deductions to be corrected and verified by comparison and consilience with experience. This general character of deductive reasoning in Sociology has been exhibited by Mill theoretically at length in his 'Logic,' and practically by repeated cautions in his 'Political Economy.' The steps of Mill are followed by almost all considerable writers upon method—Cornewall Lewis, Cairnes, Bain, Mr. Jevons in the 'Principles of Science,' Mr. Sidgwick in behalf of 'Economic Method' renouncing pretensions to precision of detail.

It cannot be expected that so terse a treatise as the present should go over ground exhausted by such writers. We must take for granted that our intelligent inquirer understands what is intelligible to the intelligent. If he believe not the authorities just cited, it would not be worth our while to resuscitate considerations long consecrated by universal acceptance. We can only consider the position of one who, understanding in a general way the nature and the need of deductive reasoning in Sociology, draws the line at deductions couched in the language of literature, refusing to employ as signs of general conceptions

[1] Cf. Plato, *Republic*, b. vi. s. 501.

mathematical symbols along with ordinary words. The theoretical weakness of this position is that there is no logical ground for drawing the line, *other than the prejudice that mathematical reasoning imports numerical data.* Such, in fact, appears to be the ground on which the objections against economical mathematics are based by Cairnes; Cairnes, whose opinion on this subject is shared by a still more distinguished analyst.[1] This prejudice having been cleared away,[2] why should not general reasonings about quantities be assisted by the *letters* appropriate to the science of quantity, as well as by ordinary *words*? 'Ego cur, acquirere pauca si possum, invideor?' the generalising genius of Mathematics unanswerably demands.

Practically, the objection *solvitur ambulando*, by the march of science which walks more securely—over the[3] flux and through the intricate—in the clear beam of mathematical intuition. The uses of this method may have been already illustrated, at least by reference to the achievements of mathematical economists. It will, however, be attempted here to present some further illustration, introduced by the conspicuous case of a country convulsed by political conspiracy and economical combination.

(I.) First as to the *political* aspect of the case has Calculus anything to teach? Nothing as to practical politics; but as to the first principles of political theory perhaps something. What is the first principle of politics? *Utilitarianism*, it would be replied by most intelligent persons of the nineteenth century, if in different terminologies, yet virtually with one accord. Of this basis what is the ground? Here we leave the visible constructions of external action descending into a subterraneous region of ultimate motives.

The motives to Political Utilitarianism are the same as in the case of Ethical Utilitarianism, some would say; and they would have to grope for a *proof of utilitarianism*, such as Mr. Sidgwick grasps at with one hand, while with the other hand he grasps the polar principle. His method proceeds by comparing

[1] *Fortnightly Review*, 1879, *Economic Method.*

[2] See pp. 2-6, and Appendix I.

[3] To treat *variables* as *constants* is the characteristic vice of the unmathematical economist. Many of the errors criticised by M. Walras are of this character. The *predeterminate Wage-fund* is a signal instance.

deductions from the utilitarian first principle with moral sentiments observed to exist; 'philosophical intuitionism' does not come to destroy common-sense, but to fulfil it, systematising it and rendering it consistent with itself. Now this method may be assisted, with regard to certain quantitative judgments of common sense, by the science of quantity;[1] proving these moral judgments to be consilient with deductions from Utilitarianism, clipping off the rough edges of unmethodical thought.

But to others it appears that moral considerations are too delicate to support the gross structure of political systems; at best a flying buttress, not the solid ground. It is divined that the pressure of *self-interest* must be brought to bear. But by what mechanism the force of self-love can be applied so as to support the structure of utilitarian politics, neither Helvetius, nor Bentham,[2] nor any deductive egoist has made clear. To expect to illuminate what Bentham has left obscure were presumptuous indeed. Yet it does seem as if the theory of the *contract-curve*[3] is calculated to throw light upon the mysterious process by which a crowd of jostling egoists tends to settle down into the utilitarian arrangement.

Thus the terms of the social contract are perhaps a little more distinctly seen to be the conditions of 'Greatest Happiness.' If the political contract between two classes of society, the landlord and the tenant class for instance, is disturbed, affected with the characteristic evil of contract 'undecidable[4] strife' and deadlock, the remedy is utilitarian legislation; as is already felt by all enlightened statesmen.

Considerations so abstract it would of course be ridiculous to fling upon the flood-tide of practical politics. But they are not perhaps out of place when we remount to the little rills of sentiment and secret springs of motive where every course of action must be originated. It is at a height of abstraction in the rarefied atmosphere of speculation that the secret springs of action take their rise, and a direction is imparted to the pure

[1] See above, pp. 76–80. And cf. the *proof of utilitarianism* in *New and Old Methods of Ethics* (by the present writer).

[2] I take the view which Mr. Sidgwick takes (*Fortnightly Review*) of Bentham's aims, and of his success.

[3] Corollary, p. 53. [4] Above, p. 29.

fountains of youthful enthusiasm whose influence will ultimately affect the broad current of events.

The province of ends is thus within the cognisance of Mathematics. What shall we say of intermediate, or proximately final, principles? The quantitative species of 'Reason is here no guide, but still a guard,' at present; and might conceivably be something more in some distant stage of evolution related to the present (agreeably to the general description of evolution) as the regularity of crystallization to the violent irregular movements of heated gas.

Let us take a question suggested, however remotely, by our heading. When 'peasant proprietorship,' 'expropriation of landlords,' and even more communistic schemes, are talked of, there are those whose way of thinking carries them on to inquire whether the level of *equality* is a thing so much to be desired *per se*, and abstracted from the expediencies of the hour, and even the age.

The demagogue, of course, will make short work of the matter, laying down some metaphysical 'rights of man.' Even Mill never quite disentangled what may be a proximate from what is the final end of utilitarianism. And it is much to be feared that a similar confusion between ends and means is entertained by those well-meaning, generally working, members of the social hive, who seem more concerned about the equilateralness of the honeycomb than the abundance of the honey. But the very essence of the Utilitarian is that he has put all practical principles in subjection, under the supreme principle. For, in that he has put all in subjection under it, he has left none that is not put under it.

How then is it possible to deduce *Equality* from 'Greatest Happiness; the symmetry of the Social Mechanism from the maximum of pleasure-energy? By mathematical reasoning such as that which was offered upon a previous page,[1] or in an earlier work,[2] such as had already been given by Bentham and the Benthamite William Thompson. Bentham, who ridicules the metaphysical rights of man and suchlike 'anarchical fallacies,'

[1] Above, p. 64.

[2] *New and Old Methods of Ethics.* The reasoning was offered in ignorance of the analogous Benthamite reasoning.

reasons down from Greatest Happiness[1] to Equality by a method strictly mathematical; even though he employ 'representative-particular' numbers[2] rather than general symbols. The argument might be made palpable by a parallel argument, constructed upon *another* of the great *arches* of exact social science, or those *concave functions*, as they might be called, in virtue of which the Calculus of Variations becomes applicable to human affairs—the *law of diminishing returns*. A given quantity of labour (and capital) will be expended most productively on a given piece of land, when it is distributed uniformly, *equally*, over the area; by a parity of reasoning which makes palpable the parity of proviso: *provided that there be no differences of quality in the ground.* If, speaking both literally and in parable, there is (indication and probability of) difference; if for the same seed and labour some ground brought forth a hundredfold, some sixtyfold, some thirtyfold, the presumption is that more should be given to the good ground.

Is there then any indication of such difference between sentients? We may not refuse once more to touch this question, however unwelcome to the modern reader; otiose to our unphilosophical aristocrats, and odious to our democratical philosophers.

(I.) First, then, it may be admitted that there is a difference with respect to *capacity for happiness* between man and the more lowly evolved animals; and that *therefore*—among or above other considerations—the interests of the lower creation are neglectible in comparison with humanity, the privilege of man is justified. Or if any so-called utilitarian, admitting the practical conclusion, refuses to admit its *sequence* from the premiss, affirming some first principle in favour of the privilege of his own species, he must be gently reminded that this affirmation of first principles not subordinate to the Utilitarian Principle is exactly what the great utilitarian called 'ipse-dixitism'; and also—in case he protests against the oligarchical

[1] Bentham apud Dumont, *Traités de Législation: Code Civil*, ch. vi.; *Principles of Pathology* (Bowring's edition), vol. i.; ib. vol. ii. 228, &c.; thus evincing a perfectly clear idea of the utilitarian end, more than might have been inferred from some of his phraseology.

[2] Often a precarious method. Cf. Marshall, *Foreign Trade*, ch. i. p. 4.

tendencies of our position—that *he*, not we, is the oligarch, the oligarchical demagogue levelling down to himself, and there drawing the line. But the pure Utilitarian, drawing no hard and fast line, according to the logical divisions of scholastic *genera* or pre-Darwinian *Real Kinds*, and admitting no ultimate ground of preference but *quantity of pleasure*, 'takes every creature in and every kind,' and 'sees with equal eye,' though he sees to be unequal, the happiness of every sentient in every stage of evolution.

(II.) Again, it may be admitted that there are differences of *capacity for work*, corresponding, for example, to differences of age, of sex, and, as statistics about wages prove, of race. It would be a strange sort of rational benevolence which in the distribution of burdens would wish to equalise the objective circumstance without regard to subjective differences.

(III.) Now (as aforesaid [1]) the admission of different relations in different individuals between external circumstances and internal feeling in the case of one species of (negative) pleasure is favourable to the admission of such differences in the case of other species of pleasure, or pleasure in general. Not only do we see no reason why the latter difference, if agreeable to observation, *ought not* to be admitted; but also we see a reason why it *has not* been admitted or not observed. For in the former case we have what in the latter case we have not, the same quantity of feeling in different individuals corresponding to different values of an external variable, namely the (neighbourhood of) the infinite value of fatigue to different external limits of work done. And everyone is acquainted with those whose physical or intellectual power he himself could not equal, 'not even if he were to burst himself;' whereas in the case of pleasure in general—owing apparently to the rarity or irregularity of the very high values of pleasure—we are reduced to the observation of different increments of pleasure occasioned by the same increment of means.

But is this observation insufficient? Or can it be indifferent to the utilitarian whether a given opportunity or increment of means is bestowed where it occasions but a single simple sensuous impression of μονόχρονος ἡδονὴ, or a pleasure truly

[1] Above, p. 59.

called 'higher,' or 'liberal,' or 'refined'—integrated by redintegrating memory, multiplied by repeated reflection from the 'polished breast' of sympathisers, in fine raised to all the powers of a scientific and a romantic imagination? Can we think it indifferent whether the former or the latter sort of sentience shall be put into play?

(IV.) Put into play, or brought into existence. For at what point shall we stop short and refuse to follow Plato while, inspired with an 'unconsciously implicit,'[1] and sometimes an explicit,[2] utilitarianism, he provides for the *happiness* (it is submitted, with due deference to Aristotle[3]), not only of the present, but of succeeding generations? Or should we be affected by the authority of Mill, conveying an impression of what other Benthamites have taught openly, that all men, if not equal, are at least *equipotential*, in virtue of equal educatability? Or not connect this impression with the more transitory parts of Mill's system: a theory of Real Kinds, more Noachian than Darwinian, a theory of knowledge which, by giving all to experience gives nothing to heredity, and, to come nearer the mark, a theory of population, which, as pointed out by Mr. Galton (insisting only on quantity of population) and, taking no account of *difference of quality*, would probably result in the ruin of the race? Shall we resign ourselves to the authority of pre-Darwinian prejudice? Or not draw for ourselves very different consequences from the Darwinian law? Or, rather, adopt the 'laws and consequences' of Mr. Galton?[4]

To sum up the powers claimed for our method: if in some distant stage of evolution there may be conceived as practicable a distinction and selection, such as Plato adumbrated in the 'Republic,' the selected characters perhaps not so dissimilar from the Platonic ideal—wise and loving, with a more modern spirit both of science and romance—but the principle of selection, not intellect so much as feeling, capacity for happiness; then the delicate[5] reasoning about capacity

[1] Mr. Sidgwick's happy phrase.

[2] Καλλιστὰ γὰρ δὴ τοῦτο λέγεται καὶ λελέξεται, ὅτι τὸ μὲν ὠφέλιμον καλὸν, τὸ δὲ βλαβερὸν αἰσχρόν.—Plato's *Republic.*

[3] *Politica*, v. [4] *Hereditary Genius;* end of penultimate chapter.

[5] Above, p. 65.

would seem to stand in need of mathematical, if not symbols, at least conceptions. And even at present it is well, at whatever distance, to contemplate the potentiality and shadow of such reasoning. For though the abstract conclusions have no direct bearing upon practical politics (for instance, extension or redistribution of suffrage), determined by more proximate utilities—just as Bentham protests that his abstract preference for equality does not militate against the institution of property—nevertheless it can hardly be doubted that the ideal reasonings would have some bearing upon the general drift and tendency of our political proclivities. And at any rate the history of all dogma shows that it is not unimportant whether a faith is held by its essential substance, or some accidental accretion. And the reasonings in question may have a use in keeping the spirit open to generality and free from prepossession, the pure ideal free from the accreting crust of dogma. From semi-à-priori 'innate perceptions' dictated by an 'analytic[1] intelligence,' from '*equity*,' and 'equalness of treatment,' and 'fairness of division;'[2] which, if they gave any distinct direction at all (other, of course, than what is given by merely utilitarian[3] considerations), would be very likely to give a wrong direction, meaning one which is opposed to the Universalistic Hedonism or Principle of Utility established by the more inductive methods of Sidgwick and of Hume. From dictates indistinct and confusing, or, if distinct—at least about a subject so amenable to prejudice as 'equalness' and 'equity'—most likely to be wrong. To show which danger it is sufficient (and it appears necessary, at a not unfelt sacrifice of deference) to observe that the same semi-à-priori method, applied to Physics, in the course of a prolonged discussion of 'Force' and its 'Persistence,' never clearly distinguishes, nay, rather confounds, 'Conservation of Momentum' and 'Conservation of Energy'! while it is distinctly stated that the law of the *inverse square* is 'not simply an empirical one, but one deducible mathematically from the relations of space—one of which the negation is inconceivable.'[4] Is it wise, is it safe, to

[1] Herbert Spencer, *Data of Ethics*, s. 62.

[2] Ib. s. 60, p. 164.

[3] Ib. ss. 68, 69, etc.

[4] Id. *First Principles*, s. 18.

weight and cramp science with à-priori dogmas such as this—in view of the possibility of a Clerk-Maxwell after all discovering, by the ordinary (Deductive) method of Inductive Logic, that there is attraction between atoms according to a law of inverse *fifth* power? An inductively deductive method in Sociology may have similar surprises for the dogmatic isocrat forthcoming; but they will certainly not come, there will not come any development, if we resign ourselves with a Byzantine sloth to à-priority or other authority more dear to the utilitarian; not dissociating the faith of love from the dogma of equity, from the accreted party-spirit and isocratic prejudice of Benthamite utilitarianism, the 'pure ethereal sense' and unmixed flame of pleasure.

And lastly, 'whether these things are so, or whether not;' about a subject so illusory, where the vanity and the very virtues of our nature, oligarchical pride, democratical passion, perturb the measurements of utility; not slight the advantage of approaching the inquiry in the calm spirit of mathematical truth.

Thus it appears that the mathematical method makes no ridiculous pretensions to authority in practical politics. There is no room for the sarcasm of Napoleon complaining that Laplace wished to govern men according to the Differential Calculus. The sense of practical genius need not take offence. The mathematical method has no place in camps or cabinets; but in a philosophic sphere in which Napoleon had neither part nor lot, and which he scouted as 'Ideology.'[1]

(II.) Let us turn now to the *economical* aspect of the case before us: *combination* of tenants against landlords, which the present crisis in Ireland[2] is thought to involve. Here also the dry light may illuminate the troubled scene of dead-locked unions; and by an unobvious path lead up again to the principle of utility as the basis[3] of arbitration. The *fair* rent is seen to be the *utilitarian* rent.[4]

[1] Bourrienne's *Memoirs*.

[2] The *Pall Mall Gazette* has persisted in regarding the agrarian as Trades Unionist outrages.

[3] Read Mr. Crompton in *Industrial Conciliation* (cf. pp. 82, 83), and realise the need of some *principle* of arbitration.

[4] Her Majesty's Commissioners of Inquiry into the working of the Land

Here it may be proper to indicate the relation which preceding considerations upon *indeterminateness of contract* are supposed by their writer to bear to the considerations recently adduced by others, in particular Mr. Cliffe Leslie[1] and Mr. Frederick Harrison,[2] concerning the irregular and accidental character of mercantile phenomena—as contrasted with what may be called perhaps the old-Ricardian view. The two sets of considerations, ours and theirs, may be mutually corroborative; but they are for the most part distinct, though they occasionally overlap. Thus Mr. C. Leslie's contention against the equality of profits, &c., in different occupations, does not form any part of these fragmentary studies; while, on the other hand, our *second* and *fourth*[3] *imperfections* have not perhaps been noticed elsewhere. Again, the imperfection of the labour market, due to the immobility of the labourer upon which Mr. Frederick Harrison in a human spirit dwells is, analytically considered, a case of our *first imperfection*.

As there is a certain relation of alliance between these considerations and those, so they may be all exposed to the same attack, namely, that the irregularities in question, though existent in fact, do not exist in tendency, tend to disappear, and therefore may be neglected by abstract science. This is a matter of fact upon which the present writer is ill-qualified to offer an opinion. But he submits that the imperfections which it has been in these pages attempted to point out in the case of cooperative association and to trace in the case of trades-unionism, do not tend to disappear, but rather to increase, in the proximate future at least. The importance of the *second* imperfection—affecting contract with regard to certain kinds of

Act of 1870, &c., having sanctioned and supposing settled a 'fair rent,' recommend that the 'unearned increment' which may accrue should, *in the absence of first principles* to determine the distribution between landlord and tenant, be divided *equally* between them. Observing that the *contract-curve* in this case is the representation of all the possible *rents* (p. 142), we have here a simple exemplification of the theory that the *basis of arbitration* is a point on the contract-curve, roughly and practically as here the *quantitative* mean, the bisection of the indeterminate reach of contract-curve, theoretically the qualitative mean the utilitarian point (p. 55).

[1] *Fortnightly Review*, Hermathena, &c. [2] *Ibid.* 1865.

[3] Pp. 46, 47.

service—might perhaps stand or fall with the importance of Mr. Cliffe Leslie's considerations upon the inequality of remunerations.[1]

Lastly, if the argument attempted in these pages concerning the indeterminateness of *contract* is as to the premisses somewhat similar to the Positivist argument, it would fain be also as to the conclusion: the necessity of settling economical differences by a moral principle—here clothed in the language more of Mill than of Comte, and disfigured by the unfortunately ugly term *Utilitarianism*, which so imperfectly suggests what it connotes. '*Vivre pour autrui.*'

Returning from this digression, let us now sift a little more accurately the light which Mathematics may shed upon *Combinations*. Compare the analysis suggested in a previous part of this work with the general account of 'Monopolies and Combinations' in 'Economics of Industry.' The conception of *indeterminateness increasing with the increase of combination* comes out perhaps a little more clearly in the mathematical analysis. To bring out the comparison, it is best to consider some particular species of combination. Here, however, occurs the difficulty that the species as presented by the text of these supplementary remarks upon method has not been much, if at all, treated by economists. Let us take, then, combinations of workmen against employers; a deviation from our subject for which the less apology is due as it is part of the purport of some coming remarks to insist on the essential unity of the different kinds of contract.

Let us consider the argument about Trades Unions contained in the 'Economics of Industry,' book iii. chapter 6, §§ 1 and 2; or rather a certain popular argument against Trades Unions strengthened by whatever it can borrow from the passage under consideration.

It is submitted with great deference, *first*, that the conclusion does not follow from the premisses, *if* the conclusion is that trades unions tend to defeat their own object, the interest of the unionists. The premiss is that the consequence of the action of Trades Unions is a continually increasing 'check to the growth,' diminution from what it would have been, of the

[1] Above, p. 47.

wages-and-profits fund, and so of the total Remuneration of operatives. But, since the utility of the operatives is a function not only of their remuneration, but their *labour*; and, though an increasing function of the remuneration, considered as explicit, is a decreasing function of the same considered as implicit in labour;[1] it does not follow that there tends to decrease that quantity which it is the object of unions to increase —the unionists' utility at each time, or rather *time-integral of utility*. Rather, it appears from the general analysis of contract that, if any effect is produced by unions, it is one beneficial to the unionists (presupposed, of course, intelligence on their part); and that, if combination is on a sufficiently large scale, an effect is likely to be produced.

But, *secondly*, the premisses are not universally true, those of the popular argument at least; for the Marshall argument keeps 'intra spem veniæ cautus.' For though it be true that the action of unionists, if they 'refuse to sell their labour except at a reserve price,' would be to diminish ultimately the Remuneration, this result would no longer hold if the unionists were to insist, not on a *rate of wages*, leaving it to the employers to buy as much or as little work as they please at that rate, but upon *other* terms of employment—a certain quantity of remuneration in return for a certain quantity of work done. If (in our terminology) they proceeded by way of *contract-*

[1] Geometrically; let an abscissa represent *time*. Let the remunerations at each time, *as they would have been*, be represented by ordinates forming a sort of hyperbola-shaped curve as to the portion of time at least with which we are concerned--from the present, far as human eye can see (not to trouble ourselves about the vertex and the asymptote). To fix the ideas, let the approximate shape be given by $\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$. Now let the series of remunerations, as it is in consequence of the action of Unions, be $\frac{(x+c)^2}{a^2} - \frac{y^2}{b'^2} - 1 = 0$ where $b' < b$, c is positive. Let the present time correspond to the point where $y' = y$; if y' be new ordinate at any point y being the old. We have then $\frac{y - y'}{y}$ the percentage of loss of remuneration continually increasing. But the end of the unionists is not the ordinates nor the area, but the hedonic integral represented by the solid contents of a certain *quasi-hyperboloid* described upon the quasi-hyperbola. From the nature of the functions of this surface it appears that the solid contents may be greater in the latter case than in the former.—Q.E.D.

curve,[1] not by way of *demand-curve*, the presumption is that their action would increase not only their utility but their remuneration.

And, *thirdly*, even if the literary method by a sort of intuition or guess-work apprehends the truth, it can hardly comprehend the whole truth. For it appears from analysis that the tendency of combinations is not only to make contract more beneficial to the unionists, but also to make it *indeterminate*; a circumstance of some interest as bringing clearly into view the necessity of a *principle of arbitration* where combinations have entered in.

The Mathematical method does not, of course, show to advantage measuring itself with the ungeometrical arguments of Mr. Marshall, himself among the first of mathematical economists, and bearing, even under the garb of literature, the arms of mathematics; which peep out in this very place ('Economics of Industry,' p. 201). A much more favourable comparison would be challenged with the popular economists, who often express themselves rather confusedly, as Mr. Morley, in an eloquent address,[2] points out. Mr. Morley's own opinion is not very directly expressed, but is presumably opposed to 'those who deny that unions can raise wages.' Now, it is submitted that this opinion, in face of the Cairnes-Marshall arguments, can only be defended by the unexpected aid of mathematical analysis. The incident may suggest, what is the burden of these pages, that human affairs have now reached a state of regular complexity necessitating the aid of mathematical analysis; and that the lights of unaided reason—though sparkling with eloquence and glowing with public spirit—are but a precarious guide unless a sterner science fortify the way.

But what is all this to *landlords and tenants?* Or can your scanty analysis of combination in general be securely extended to the peculiar case of rent? The reply is: Yes; the reasoning about the tendency of combination to produce indeterminateness can with sufficient safety—by a sort of mathematical reduction—be extended from the general to a particular case. Symbols are not to be multiplied beyond necessity. Rather the mathematical psychist should be on his guard to

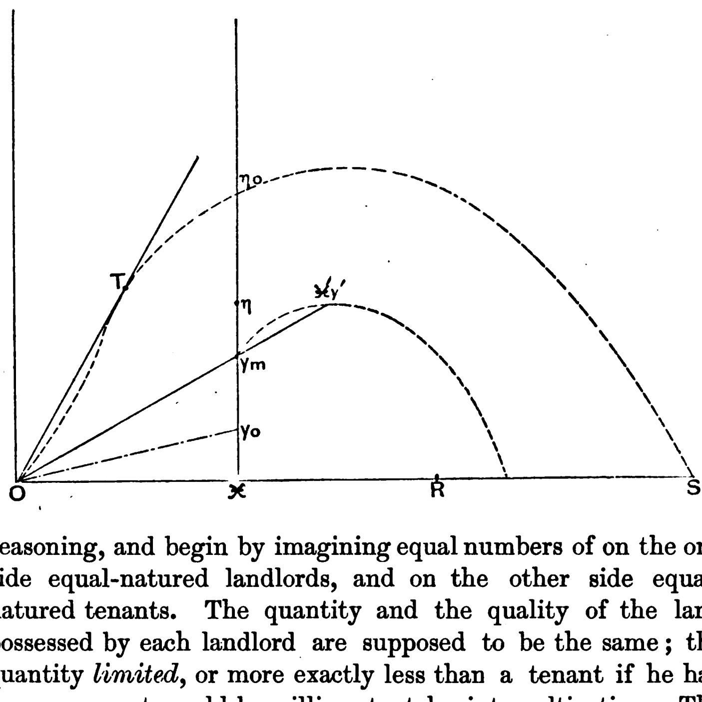
[1] See pp. 48, 116. [2] *Fortnightly Review*, 1877, p. 401.

Deduct what is but vanity or dress,
Or learning's luxury, or idleness :
Mere tricks to show the stretch of human brain.

To show, however, this very thing, the substantial unity of the theory of *contract* (whatever the *articles*), and also to further illustrate the general theory, let us attempt an analysis of the contract between landlord and cottier-tenant. We may abstract all the complications of commerce, and suppose the *competitive field* to consist only of landlords and cottier-tenants.

Let us start, then, upon the lines of previous trains of

FIG. 6.



reasoning, and begin by imagining equal numbers of on the one side equal-natured landlords, and on the other side equal-natured tenants. The quantity and the quality of the land possessed by each landlord are supposed to be the same; the quantity *limited*, or more exactly less than a tenant if he had to pay no rent would be willing to take into cultivation. The requirements and capacities of the tenants likewise are for the moment supposed equal. Let us represent the portion of land owned by the landlord as a portion of the abscissa $o\,x$, and the corresponding rent paid by a length measured along the other co-ordinate. And let us proceed to write down in this particu-

lar case the functions whose general character has been already described.

P, the *utility-function* of X the landlord, is $F(y)$ (subject to a certain discontinuity which will be presently suggested). Π, the *utility-function* of Y the tenant, is

$$\Phi(\phi(e)x - y) - \psi(xe)$$

subject to the condition $\left(\frac{d\Pi}{de}\right) = 0$. Here Φ as before is a pleasure-function. e is the amount of *objective-labour* (muscular energy or other objective measure of labour) put forth by Y, *per unit of land*. $\phi(e)$ is the corresponding *produce per unit*; a function which, according to the *law of diminishing returns*, has its first differential continually positive, and its second differential continually negative. xe is the total objective labour. $\psi(xe)$ the corresponding *subjective labour*, or disutility; a function which according to the *law of increasing fatigue* has both its first and second differential continually *positive*. Since e is variable at the pleasure of Y, he will vary it (whatever x may be), so that his utility as far as in him lies may be a maximum; whence $\left(\frac{d\Pi}{de}\right) = 0$. Let us for convenience designate the function which results from the indicated elimination of e by $\pi(xy)$.

The *indifference-curves* of the landlord if he have no other use for his land are horizontal lines; importing that it is indifferent to the landlord how much land he lets, provided he gets the same (total) rent. Let us however for the sake of illustration, and indeed as more real, suppose that the landlord can always make sure of a certain minimum, by employing his land otherwise, e.g. not letting it to cottier cultivators, but to capitalist graziers. If then the landlord's income from lands thus otherwise employed be proportionate to the land thus employed at a certain *rate* per unit of land, the landlord's *indifference-curve* may be represented by Oy_0 and parallel lines (Fig. 6).

The *indifference-curves* of the tenant are given by the differential equation $\left(\frac{d\pi}{dx}\right)dx + \left(\frac{d\pi}{dy}\right)dy = 0$. Now $\left(\frac{d\pi}{dx}\right)$ is by hypothesis *positive* in the neighbourhood of the

origin, and negative ultimately; since $0\,x$ has been assumed less than the quantity of land which Y would be willing to take into cultivation *without rent*, which quantity is given by the equation $\left(\frac{d}{dx}\right)\pi(x,o)-=0$. And $\left(\frac{d\,\pi}{d\,y}\right)=\left(\frac{d\Pi}{d\,y}\right)=-\Phi'(x\phi(e)-y)$ is essentially *negative*. Thus the indifference curve ascends in the neighbourhood of the origin and descends as indicated in the figure to the point R where $\left(\frac{d}{d\,x}\right)\pi\,(x,\,o)=0$. Again,

$$\frac{d_2\,y}{d\,x^2}=\frac{\left(\frac{d\,\pi}{d\,x}\right)^2\left(\frac{d_2\,\pi}{d\,y^2}\right)-2\,\frac{d\,\pi}{d\,x}\,\frac{d\,\pi}{d\,y}\left(\frac{d_2\,\pi}{d\,x\,d\,y}\right)+\left(\frac{d\,\pi}{d\,y}\right)^2\left(\frac{d_2\,\pi}{d\,x^2}\right)}{-\left(\frac{d\,\pi}{d\,y}\right)^3}$$

where $\left(\frac{d_2\,\pi}{d\,x^2}\right)=\left(\frac{d_2\Pi}{d\,x^2}\right)+2\left(\frac{d_2\,\Pi}{d\,x\;d\,e}\right)\frac{d\,e}{d\,x}+\left(\frac{d^2\Pi}{d\,e^2}\right)\left(\frac{d\,e}{dx}\right)^2+\cdot$ $\left(\frac{d\;\Pi}{d\,e}\right)\frac{d_2\,e}{d\,x^2}$, the last term being equal to zero in virtue of the equation $\left(\frac{d\Pi}{d\,e}\right)=0$. And $\frac{d\,e}{dx}=\frac{\left(\frac{d_2\,\Pi}{d\,e\,d\,x}\right)}{-\left(\frac{d_2\Pi}{d\,e^2}\right)}$. And similarly for the other second differentials of little π. Working out the somewhat elephantine formulæ thus indicated, and attending to the character of the functions $\Phi\ \phi\ \psi$, we should find that [1] the curve is convex when $\frac{d\,y}{d\,x}$ is negative. The attention of the student is directed to this, if expanded rather lengthy, *mathematical reasoning, for which never a numerical datum is postulated, about a social subject.* The curves may be (I think) convex at starting. Thus in figure 6, $o\,T\,\eta_0\,s$ is a fair representation of Y's *indifference-curve* through the origin. The curve through y_m and $(x'\,y')$ represents (part of) another member of the same family.

The *demand-curve* of the landlord is the ordinate at the point x from above the point y_0. The landlord will be willing to take *any* amount of rent for his land above that minimum! Or, in other terms, the quantity of land which he offers at any

[1] Compare the reasoning at pp. 35, 36.

rate of rent (indicated by the angle between a vector and the abscissa) is $o\,x$. The demand-curve of the tenant is the locus of points of contact between vectors drawn from the origin and *indifference-curves*. In the figure it is supposed to pass through T, η, and R; the last point indicating the quantity of land demanded by the tenant at rate of rent zero.

So far as to what may be called *personal* or individualistic functions. What of the *mutual* function, which plays so large a part in our speculations, the *contract-curve*? The available portion of the contract-curve is $y_0\,\eta_0$, the portion of the ordinate at x intercepted between the indifference-curves from the origin. For it is easy to see that if the index be placed anywhere to the left (it cannot by hypothesis be placed on the right) of this line it will run down under the force of concurrent self-interests to the line in question. For instance, at the point T, the indifference-curve of Y is drawn in the figure, and the indifference-curve of X is a line parallel to $0y_0$; between which and the corresponding lines at each point the index will continually move down to the line $x\,\eta_0$ (assuming at least a certain limitation or relative smallness of $o\,x$). Here, however, occurs the interesting difficulty that the general condition $\dfrac{d\,\mathrm{P}}{d\,x}\dfrac{d\,\Pi}{d\,y} - \dfrac{d\,\Pi}{d\,x}\dfrac{d\,\mathrm{P}}{d\,y} = 0$ is not satisfied by the line $y_0\,\eta_0$. What is the rationale of this? It may be thus stated. The *contract-curve* expresses the condition of a certain hedonic (relative) *maximum*. Now the condition of this maximum is in general, according to the general principles of the Calculus of Variations, the vanishing of a certain first term of variation. But the general rule of the Calculus of Variations is suspended in particular cases of *imposed conditions*; according to a principle discovered by Mr. Todhunter, which is probably of the greatest importance in the calculus as applied to human affairs. Now the case before us of quantity of land *fixed* and *small* constitutes such an imposed condition and barrier as is presented in so many of Mr. Todhunter's problems. In the metaphorical language already employed,[1] we might conceive the contractors' joint-team driven over the plain up to the barrier $y_0\,\eta_0$; ready to move on to the right of

[1] Above, p. 24.

the line if the barrier were removed, but incapable of moving either up or down the line. If the quantity of land were *fluent*, as in general *articles* of contract are to be regarded, then the ordinary form of the contract-curve will reappear. That the quantity of land should be regarded as fluent it is not necessary that it should be absolutely unlimited, as in general articles of contract have a superior limit *e.g.*, the quantity of labour a man can offer. It suffices that the quantity of land should be large; more exactly that the angles made by the indifference-curves of Y at each point of the ordinate with the direction $o\,x$ should be greater than the angles made by the indifference-curves of X.

Let us now proceed to investigate the *final settlements* in the *field of competition* just described. The first condition[1] of a final settlement is that the whole field be collected at a point on the contract-curve. The second condition is that recontract be impossible. What then are those points at which the whole field being concentrated recontract is possible? Those at which p landlords can recontract[2] with q tenants.

By definition of contract-curve p and q are unequal. The recontract, or at least the *settlements* to one of which it tends, may be represented by a *supplementary* contract-system constructed on the analogy of that above[3] indicated. A little attention will show that p must be greater than q when the point y_o falls as in the figure below the point η to be presently defined. The *supplementary* system then consists of the original contract-curve and a perpendicular to the abscissa at the point x' such that $p \times o\,x = q \times o\,x'$; and it imports that the recontractors tend to the following arrangement: the p landlords on a point, say $x\,y$, of the original contract-curve, and the q tenants on a point $x'y'$ determined by the intersection of a vector through $x\,y$, with the supplementary contract-curve or perpendicular at x'. Accordingly, if as just supposed the whole field is concentrated at a point $x\,y$ on the contract-curve p landlords can[4] recontract with q tenants so long as y

[1] Above, p. 35.

[2] Each recontracting for himself, of course, the fourth imperfection being not in general presupposed. [3] P. 37.

[4] It may be a nice question how far, as a matter of fact, the process of

is such that the corresponding point $x' y'$ falls within the tenant's indifference-curve drawn through $x y$. The recontract will just be impossible when $x' y'$ is on the intersection of the indifference and supplementary curves. It will appear that the larger is the fraction $\frac{p}{q}$ the longer, as we ascend the contract-curve moving from y_o, is impossibility of recontract deferred. The last point, therefore, at which recontract is possible, is y_m, the (tenant's) indifference-curve through which meets the vector from the origin on the ordinate at x', where $(m-1)\, o x' = m\, o x$. The points beyond y_m are *final settlements*.

By parity it may be shown that the points on the contract-curve in the neighbourhood of η_o are not *final* settlements; but that the system if placed at any of them will move away under the influence of *competition between landlords*; on to a point η_m, the indifference-curve through which meets the vector from the origin on the ordinate at x'' where $m\, o x'' = (m-1)\, o x$.

Between η_m and y_m there is a reach of contract-curve consisting of *final settlements. The larger m is the smaller, is the reach of indeterminate contract.*

It is clear that similar reasoning will hold if we suppose our landlords and tenants to be not individuals, but equal corporate competitive units, in short, equal *combinations* as in these pages understood. Thus it is clearly seen how the increase of combination tends to increase indeterminateness in a sense favourable to the combiners.

Clearly seen in the abstract; and what has been sighted in the abstract will not be lost sight of as it becomes immersed in the concrete: when we suppose the numbers of the parties on each side, the natures of the tenants, the quantities and qualities of land, the size of combinations, &c., to be *unequal.*

recontract in imperfect competition will involve the conception of *rate of exchange*—the tenant for instance endeavouring to vary any existing contract—because at the *rate* presented by that contract, the ratio of the articles exchanged, he would be willing to take, he *demands*, more land. It has seemed best in treating of contract in general to keep clear of a conception which is, it is submitted, *essential* only to one species of contract, that determined by perfect competition.

The treatment of different numbers on each side is suggested by the theory of the supplementary contract-curve. The treatment of different natures may be thus indicated in the important instance when the numbers on each side are indefinitely large. In this instance, it may be premised, upon the supposition of equality the points η_m and y_m coincide at the point η, where the vector from the origin touches the (tenant's) indifference-curve on the contract-curve, and which is accordingly on the tenant's *demand-curve.*[1] And it is also on the landlord's demand-curve.[2] And thus *contract is determined by the intersection of the demand-curves.* Here we suppose all the tenants to have the same requirements, the same indifference-curves. We might conceive the perfectly similar curves which are touched at η coincidently heaped up. Now, the natures varying, let the curves no longer identical slide away from each other, still keeping in contact with the itself-moving vector; subject to the condition that the sum of the lands let is equal to the sum of the lands rented. Or more precisely: subject to the said condition, draw a vector from the origin such that it touches a member of every family of (tenant's indifference) curves. It is clear that equilibrium is then attained. No tenant wants any more land at the rate of rent indicated by the vector, and therefore does not, as he otherwise would, tend to raise the rate in order to obtain more land at the same, or even a slightly increased, rate. And no landlord has an *effective* demand for more rent, since he has no more land.

The preceding investigation applies to the case of different quantities of land. The case of different qualities is one which has not been explicitly treated in these pages. But its treatment is suggested by analogy. If, for instance, there are two species of land, x and y, the rent being represented Z ($=Z_x+Z_y$), the *contract-locus* might be regarded as a *curve of double curvature,* down which—down from their maximum utility—the tenants are worked by competition, the further as they are less combined. It would be easy, were it relevant, to contemplate from this point of view the Ricardo-Mill theory of the 'worst land paying no rent,' &c.

[1] See Index *sub voce* Demand-curve. [2] Above, p. 141.

With regard to *combinations* in the concrete, it may be observed that, while in the abstract symmetrical case equality of distribution between combiners might be taken for granted, we must in case of unequal natures presuppose in general a *principle of distribution* as an *article of contract between members of a combination*; presumably tending to the utilitarian distribution.

It was not promised that this final efflorescence of analysis would yield much additional fruit, though perhaps one who knew where to look might find some slight vintage. Attention may be directed to the possible initial convexity of the tenant's indifference-curve. It will depend upon the presence or absence of this property whether or not the tenant can be deprived by competition of the *entire utility* of his bargain in *perfect* competition; and the same property presents interesting peculiarities in the case of imperfect competition.

What it has been sought to bring clearly into view is the essential identity (in the midst of diversity of *fields* and *articles*) of *contract*; a sort of unification likely to be distasteful to those excellent persons who are always dividing the One into the Many, but do not appear very ready to subsume the Many under the One.

Mr. Cliffe Leslie is continually telling us that nothing is to be got from such abstractions as the 'desire of wealth and aversion for labour,' feelings different in different persons, and so forth. Yet he would surely admit that there is a general theory of contract, of the bargain between individuals actuated by those abstract desires, irrespective of the diversity of their tastes,[1] and all the information about particulars which Mr. Cliffe Leslie desiderates. Thus confining our attention to the simple case of two [2] sets of contractors, Xs and Ys—it may be Producers and Consumers, Employers and Employed, Lenders and Borrowers, Landlords and Tenants, International traders; prescinding this simple case for convenience of enunciation, we might write down I think some such (not the most general, but quite generalisable) *laws of contract*—contract qualified by competition.

I. Where the numbers on both sides are indefinitely large,

[1] See p. 145. [2] See above, p. 17.

and there are no *combinations*, and competition is in other respects *perfect*, contract is determinate.

II. Where competition is imperfect, contract is indeterminate.

III. *Cœteris paribus*, if the numbers on one side are decreased (or increased) each of the (original) members on that side, in perfect competition gains in point of utility (or loses); in imperfect competition *stands* [1] *to gain* (or *stands to lose*).

IV. In perfect competition, if, *cœteris paribus*, the supply on one side—meaning the amount of article offered at each price —if this whole scale of offers is increased on one side, whether from increase of numbers on that side or otherwise, then the other side gains; and an analogous proposition is true of imperfect competition.

The last two theorems have important exceptions mostly requiring mathematical analysis for their investigation; those, for instance, which may be presented by Mr. Marshall's *second* class of curves (if the introduced change might cause a jump from the neighbourhood of the first intersection of demand-curves to that of the third).

The preceding and the many similar abstract theorems are important as well as those historical inquiries on which Mr. Leslie [2] lays so much stress. It suffices to say that on a form of the third theorem J. S. Mill propounded his counsels to the wage-earning classes, and shaped and re-shaped the policy of millions upon a theory of capital-supply, at first affected with what may perhaps be called the special [3] vice of unsymbolical Economics, at length [4] corrected, and after all [5] imperfectly because ungeometrically apprehended.

It is easy with Cairnes protesting against the identification of Labour with commodities to say: [6] 'Verbal generalizations are of course easy,' and the equation of Demand to Supply is 'what any costermonger will tell you.' But the noble costermonger would not perhaps find it so easy to tell us about Mr. Marshall's Demand-curves *Class II.*, or other exceptional cases,

[1] See p. 43.

[2] There is room for all, as Prof. Jevons points out in a temperate article in the *Fortnightly Review*.

[3] Above, p. 127.

[4] *Review of Thornton.*

[5] Above, p. 5.

[6] *Leading Principles*, Part II. ch. i. § 2.